Solution to Practice 2m

B4(a) Every vector in the plane has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 - x_2 \end{bmatrix}$, so to see whether or not $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ is a basis for the plane, we will row reduce the following augmented matrix to row echelon form.

$$\left[\begin{array}{cc|c} -1 & 1 & x_1 \\ 1 & 1 & x_2 \\ 1 & -3 & -2x_1 - x_2 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{cc|c} -1 & 1 & x_1 \\ 0 & 2 & x_1 + x_2 \\ 0 & -2 & -x_1 - x_2 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \end{array}$$
$$\sim \left[\begin{array}{cc|c} -1 & 1 & x_1 \\ 0 & 2 & x_1 + x_2 \\ 0 & 0 & 0 \end{array}\right]$$

From our row echelon form, we see that the COEFFICIENT matrix has rank 2, so our set of 2 vectors is linearly independent. Moreover, since the REF of the AUGMENTED matrix does not have a bad row, we see that our set of vectors spans the plane. Thus, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ is a basis for the plane.

B4(b) Every vector in the plane has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$, so to see whether or not $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for the plane, we will row reduce the following augmented matrix to row echelon form.

$$\left[\begin{array}{cc|c} 1 & 1 & x_1 \\ -2 & 0 & x_2 \\ 1 & 4 & 4x_+ 2x_2 \end{array}\right] \begin{array}{c} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 2 & 2x_1 + x_2 \\ 0 & 3 & 3x_1 + 2x_2 \end{array}\right] \begin{array}{c} \\ \\ R_3 - (3/2)R_2 \end{array}$$
$$\sim \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 2 & 2x_1 + x_2 \\ 0 & 0 & (1/2)x_2 \end{array}\right]$$

From our row echelon form, we see that the REF of the augmented matrix has a bad row, so our set does not span the plane. As such, $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}$ is not a basis for the plane.

B4(c) Every vector in the hyperplane has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ (2/5)x_1 + (3/5)x_2 \end{bmatrix}$, so to see whether or not $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the hyperplane, we will row reduce the following augmented matrix to row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 1 & -1 & 0 & x_2 \\ 0 & 1 & 1 & x_3 \\ 1 & 0 & 0 & (2/5)x_1 + (3/5)x_2 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & -2 & 0 & -x_1 + x_2 \\ 0 & 1 & 1 & x_3 \\ 0 & -1 & 0 & -(3/5)x_1 + (3/5)x_2 \end{array}\right] \begin{array}{c} \\ \\ R_3 + (1/2)R_2 \\ R_4 + (1/2)R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & -2 & 0 & -x_1 + x_2 \\ 0 & 0 & 2 & -(1/2)x_1 + (1/2)x_2 + x_3 \\ 0 & 0 & 0 & (-11/10)x_1 + (11/10)x_2 \end{array}\right]$$

From our row echelon form, we see that the REF of the augmented matrix has a bad row, so our set does not span the plane. As such, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is not a basis for the hyperplane.

B4(d) Every vector in the hyperplane has the form $\vec{x} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, so to see whether or not $\left\{ \begin{bmatrix} 6 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the hyperplane, we will row reduce the following augmented matrix to row echelon form.

$$\left[\begin{array}{ccc|c} 6 & 0 & -2 & -2x_3 \\ 1 & 3 & 0 & x_2 \\ -3 & 0 & 1 & x_3 \\ 0 & 1 & 1 & x_4 \end{array}\right] R_3 + (1/2)R_1 \sim \left[\begin{array}{ccc|c} 6 & 0 & -2 & -2x_3 \\ 1 & 3 & 0 & x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & x_4 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ R_3 \updownarrow R_4 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & x_2 \\ 6 & 0 & -2 & -2x_3 \\ 0 & 1 & 1 & x_4 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 - 6R_1 \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & x_2 \\ 0 & -18 & -2 & -6x_2 - 2x_3 \\ 0 & 1 & 1 & x_4 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 + (1/18)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & x_2 \\ 0 & -18 & -2 & -6x_2 - 2x_3 \\ 0 & 0 & 8/9 & -(1/3)x_2 - (1/9)x_3 + x_4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our row echelon form, we see that the COEFFICIENT matrix has rank 3, so our set of 3 vectors is linearly independent. Moreover, since the REF of the AUGMENTED matrix does not have a bad row, we see that our set of vectors spans the hyperplane. Thus, $\left\{ \begin{bmatrix} 6 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the hyperplane.

B7(a) To see if this set is a basis for $\mathbb{R}^3$, we need to find the rank of the following matrix:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 3 & -1 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -7 & -7 \end{bmatrix} \begin{matrix} \\ \\ R_3 + (7/3)R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

From the row echelon form, we see that the rank of the coefficient matrix is 2, and not 3, so the given set is not a basis for $\mathbb{R}^3$.

B7(b) The given set has 4 vectors, but only sets with three vectors can be a basis for $\mathbb{R}^3$, so we know that the given set is not a basis for $\mathbb{R}^3$.

B7(c) To see if this set is a basis for $\mathbb{R}^3$, we need to find the rank of the following matrix:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 7 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 6 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix}$$

From the row echelon form, we see that the rank of the coefficient matrix is 3. Thus, by Theorem 5, the given set is a basis for $\mathbb{R}^3$.

B7(d) The given set has 2 vectors, but only sets with three vectors can be a basis for $\mathbb{R}^3$, so we know that the given set is not a basis for $\mathbb{R}^3$.