Solution to Practice 2l

B5(a) To determine whether the given set is linearly independent, we need to look at the solution to the following homogeneous linear system:

$$\begin{array}{ccccccc} t_1 & + & t_2 & & & = & 0 \\ & & 2t_2 & + & t_3 & = & 0 \\ t_1 & & & + & 2t_3 & = & 0 \\ & & t_2 & + & 3t_3 & = & 0 \end{array}$$

And in order to find the solution, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_1 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{array}{c} \\ R_2 \updownarrow R_4 \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{array}{c} \\ \\ R_3 + R_2 \\ R_4 - 2R_2 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & -5 \end{bmatrix} \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in row echelon form, and by looking at it we see that the rank of the coefficient matrix is 3. As this is the same as the number of vectors, by Lemma 3 our set is linearly independent.

B5(b) To determine whether the given set is linearly independent, we need to look at the solution to the following homogeneous linear system:

$$\begin{array}{ccccccccc} t_1 & + & t_2 & & & + & t_4 & = & 0 \\ & & 2t_2 & + & t_3 & - & 3t_4 & = & 0 \\ t_1 & & & + & t_3 & + & t_4 & = & 0 \\ t_1 & & & + & 2t_3 & & & = & 0 \\ & & t_2 & - & 2t_3 & + & t_4 & = & 0 \end{array}$$

And in order to find the solution, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_1 \\ R_4 - R_1 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{array}{c} \\ R_2 \updownarrow R_5 \\ \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 1 & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \\ R_5 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ R_5 + 5R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in row echelon form, and by looking at it we see that the rank of the coefficient matrix is 3. Since this is less than the number of vectors, we know that our set is linearly dependent. As such, we must now find all linear combinations of the vectors that make $\vec{0}$. But that simply means completing the steps to find the solution to our system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ -R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_3 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Putting the RREF matrix into equations, we have

$$\begin{array}{ccccccc} t_1 & & & + & 2t_4 & = & 0 \\ & t_2 & & - & t_4 & = & 0 \\ & & t_3 & - & t_4 & = & 0 \end{array}$$

and replacing the variable t_4 with the parameter t, we have

$$\begin{array}{ccccccc} t_1 & & & + & 2t & = & 0 \\ & t_2 & & - & t & = & 0 \\ & & t_3 & - & t & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \\ t \end{bmatrix}$$

As such, we see that

$$-2t\begin{bmatrix}1\\0\\1\\1\\0\end{bmatrix} - t\begin{bmatrix}1\\2\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\1\\2\\-2\end{bmatrix} + t\begin{bmatrix}1\\-3\\1\\0\\1\end{bmatrix},\ t \in \mathbb{R}$$

is the list of all linear combinations of the vectors that are $\vec{0}$.

B6(a) First, we need to row reduce the matrix whose columns are our vectors:

$$\begin{bmatrix}1 & -1 & 3\\1 & 1 & 1\\2 & -1 & 5\\1 & 2 & k\end{bmatrix}\begin{matrix}\\R_2-R_1\\R_3-2R_1\\R_4-R_1\end{matrix} \sim \begin{bmatrix}1 & -1 & 3\\0 & 2 & -2\\0 & 1 & -1\\0 & 3 & k-3\end{bmatrix}\begin{matrix}\\(1/2)R_2\\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix}1 & -1 & 3\\0 & 1 & -1\\0 & 1 & -1\\0 & 3 & k-3\end{bmatrix}\begin{matrix}\\ \\R_3-R_2\\R_4-3R_2\end{matrix} \sim \begin{bmatrix}1 & -1 & 3\\0 & 1 & -1\\0 & 0 & 0\\0 & 0 & k\end{bmatrix} R_3 \updownarrow R_4$$

$$\sim \begin{bmatrix}1 & -1 & 3\\0 & 1 & -1\\0 & 0 & k\\0 & 0 & 0\end{bmatrix}$$

This last matrix is in row echelon form. In order for our set of vectors to be linearly independent, we need the rank of our matrix to be 3. This means that we need row 3 to have a pivot in it, which means that we need $k \neq 0$. So, our set of vectors is linearly independent for all values of k EXCEPT $k = 0$.

B6(a) First, we need to row reduce the matrix whose columns are our vectors:

$$\begin{bmatrix}1 & -1 & -1\\-1 & 1 & 1\\3 & 2 & k\\1 & 1 & 5\end{bmatrix}\begin{matrix}\\R_2+R_1\\R_3-3R_1\\R_4-R_1\end{matrix} \sim \begin{bmatrix}1 & -1 & -1\\0 & 0 & 0\\0 & 5 & k+3\\0 & 2 & 6\end{bmatrix}\begin{matrix}\\R_2 \updownarrow R_4\\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix}1 & -1 & -1\\0 & 2 & 6\\0 & 5 & k+3\\0 & 0 & 0\end{bmatrix} R_3-(5/2)R_2 \sim \begin{bmatrix}1 & -1 & -1\\0 & 2 & 6\\0 & 0 & k-12\\0 & 0 & 0\end{bmatrix}$$

This last matrix is in row echelon form. In order for our set of vectors to be linearly independent, we need the rank of our matrix to be 3. This means that we need row 3 to have a pivot in it, which means that we need $k - 12 \neq 0$. So, our set of vectors is linearly independent for all values of k EXCEPT $k = 12$.