Solution to Practice 2k

B1(a) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & -6 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & -6 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our final RREF matrix, we see that $x_1 = 2$, $x_2 = -3$, and $x_3 = -2$ is a solution to our system. As such, we have that

$$\begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

B1(b) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & -2 & 2 & -6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

At this point, our last row is a bad row, so we know that the system is inconsistent. That is, we know that we will not be able to find x_1, x_2 and x_3 to satisfy our equations. Thus, $\begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ is not in Span$B$.

B1(c) We are trying to find x_1, x_2, and x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

This is the same as a system of four equations in three unknowns, with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right]$$

To solve this, we will row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & -2 & 2 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ -R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From our final RREF matrix, we see that $x_1 = 3$, $x_2 = 0$, and $x_3 = -2$ is a solution to our system. As such, we have that

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

B3(a) $\left[\begin{array}{cc|c} 1 & 1 & x_1 \\ -1 & 2 & x_2 \\ 2 & 5 & x_3 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 3 & x_1 + x_2 \\ 0 & 3 & -2x_1 + x_3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array}$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & x_1 \\ 0 & 3 & x_1 + x_2 \\ 0 & 0 & -3x_1 - x_2 + x_3 \end{array}\right]$$

From this row echelon form matrix, we see that the system is consistent if and only if $-3x_1 - x_2 + x_3 = 0$.

B3(d) $\left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 3 & -1 & x_2 \\ 0 & 3 & x_3 \\ -2 & 4 & x_4 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \\ \\ R_4 + 2R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 0 & -1 & -3x_1 + x_2 \\ 0 & 3 & x_3 \\ 0 & 4 & 2x_1 + x_4 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 3R_2 \\ R_4 + 4R_2 \end{array}$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 0 & -1 & -3x_1 + x_2 \\ 0 & 0 & -9x_1 + 3x_2 + x_3 \\ 0 & 0 & -10x_1 + 4x_2 + x_4 \end{array}\right]$$

We see from this last matrix that the system is consistent if and only if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution to the homogeneous system

$$\begin{array}{rcrcrcrcl} -9x_1 & + & 3x_2 & + & x_3 & & & = & 0 \\ -10x_1 & + & 4x_2 & & & + & x_4 & = & 0 \end{array}$$

B3(e)
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 1 & 3 & -2 & x_2 \\ 1 & 3 & -5 & x_3 \\ -1 & 1 & 2 & x_4 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 + R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 0 & 4 & -2 & -x_1 + x_2 \\ 0 & 4 & -5 & -x_1 + x_3 \\ 0 & 0 & 2 & x_1 + x_4 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 0 & 4 & -2 & -x_1 + x_2 \\ 0 & 0 & -3 & -x_2 + x_3 \\ 0 & 0 & 2 & x_1 + x_4 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 \\ 3R_4 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 0 & 4 & -2 & -x_1 + x_2 \\ 0 & 0 & -6 & -2x_2 + 2x_3 \\ 0 & 0 & 6 & 3x_1 + 3x_4 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 0 & 4 & -2 & -x_1 + x_2 \\ 0 & 0 & -6 & -2x_2 + 2x_3 \\ 0 & 0 & 0 & 3x_1 - 2x_2 - 2x_3 + 3x_4 \end{array}\right]$$

From this row echelon form matrix, we see that the system is consistent if and only if $3x_1 - 2x_2 - 2x_3 + 3x_4 = 0$.