Solution to Practice 2j

B2(a) Number of parameters = number of variables - rank. In this case, we get that the number of parameters is 4-2=2. Turning the matrix into equations, we get

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & & & - & x_4 & = & 0 \\ & & & & x_3 & + & 2x_4 & = & 0 \end{array}$$

We need to replace the variable x_2 with the parameter s, and the variable x_4 with the parameter t. This gives us

$$\begin{array}{rcrcrcrcl} x_1 & + & 3s & & & - & t & = & 0 \\ & & & & x_3 & + & 2t & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s+t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

B2(b) Number of parameters = number of variables - rank. In this case, we get that the number of parameters is 5-3=2. Turning the matrix into equations, we get

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & - & 2x_4 & & & = & 0 \\ & & & & x_3 & + & x_4 & & & = & 0 \\ & & & & & & & & x_5 & = & 0 \end{array}$$

We need to replace the variable x_2 with the parameter s, and the variable x_4 with the parameter t. This gives us

$$\begin{array}{rcrcrcrcrcl} x_1 & + & s & & & - & 2t & & & = & 0 \\ & & & & x_3 & + & t & & & = & 0 \\ & & & & & & & & x_5 & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+2t \\ s \\ -t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

B2(c) Number of parameters = number of variables - rank. In this case, we get that the number of parameters is 6-3=6. Turning the matrix into equations, we get

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & & & & & - & 3x_6 & = & 0 \\ & & & x_3 & - & 5x_4 & & + & 4x_6 & = & 0 \\ & & & & & & x_5 & + & x_6 & = & 0 \end{array}$$

We need to replace the variable x_2 with the parameter r, the variable x_4 with the parameter s, and the variable x_6 with the parameter t. This gives us

$$\begin{array}{ccccccccccc} x_1 & + & 2r & & & & & - & 3t & = & 0 \\ & & & x_3 & - & 5s & & + & 4t & = & 0 \\ & & & & & & x_5 & + & t & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2r+3t \\ r \\ 5s-4t \\ s \\ -t \\ t \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

B3(a) The coefficient matrix for this system is

$$\begin{bmatrix} 1 & 5 & -3 \\ 3 & 5 & -9 \\ 1 & 1 & -3 \end{bmatrix}$$

To determine the rank and number of parameters, we need to row reduce to row echelon form:

$$\begin{bmatrix} 1 & 5 & -3 \\ 3 & 5 & -9 \\ 1 & 1 & -3 \end{bmatrix} \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -10 & 0 \\ 0 & -4 & 0 \end{bmatrix} R_3 - (2/5)R_2$$

$$\sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the row echelon form has two pivots, so the rank of the matrix is 2, and the number of parameters in the general solution is 3-2=1. To determine the general solution, I will row reduce our matrix the rest of the way to reduced row echelon form.

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ (-1/10)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 5R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{array}{rcrcrcl} x_1 & & & - & 3x_3 & = & 0 \\ & & x_2 & & & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter s, we get

$$\begin{array}{rcrcrcl} x_1 & & & - & 3s & = & 0 \\ & & x_2 & & & = & 0 \end{array}$$

From this we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

B3(b) The coefficient matrix for this system is

$$\begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & -3 \\ 4 & 8 & -7 \end{bmatrix}$$

To determine the rank and number of parameters, we need to row reduce to row echelon form:

$$\begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & -3 \\ 4 & 8 & -7 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -8 & 1 \\ 0 & -8 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & -8 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the row echelon form has two pivots, so the rank of the matrix is 2, and the number of parameters in the general solution is 3-2=1. To determine the general solution, I will row reduce our matrix the rest of the way to reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -2 \\ 0 & -8 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ (-1/8)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & 1 & -1/8 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 4R_2 \\ \\ \\ \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -1/8 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{array}{rcrcl} x_1 & & - & (3/2)x_3 & = & 0 \\ & x_2 & - & (1/8)x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter s, we get

$$\begin{array}{rcrcl} x_1 & & - & (3/2)s & = & 0 \\ & x_2 & - & (1/8)s & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (3/2)s \\ (1/8)s \\ s \end{bmatrix} = s \begin{bmatrix} 3/2 \\ 1/8 \\ 1 \end{bmatrix}$$

B3(c) The coefficient matrix for this system is

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & 7 & 0 & -14 \\ 1 & 3 & 0 & -6 \\ 1 & 4 & 0 & -8 \end{bmatrix}$$

To determine the rank and number of parameters, we need to row reduce to row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 2 & 7 & 0 & -14 \\ 1 & 3 & 0 & -6 \\ 1 & 4 & 0 & -8 \end{bmatrix} \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 5 & -2 & -10 \\ 0 & 2 & -1 & 4 \\ 0 & 3 & -1 & -6 \end{bmatrix} (1/5)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2/5 & -2 \\ 0 & 2 & -1 & 4 \\ 0 & 3 & -1 & -6 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2/5 & -2 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & -2/5 & -2 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the row echelon form has three pivots, so the rank of the matrix is 3, and the number of parameters in the general solution is 4-3=1. To determine the general solution, I will row reduce our matrix the rest of the way to reduced row echelon form.

$$\left[\begin{array}{cccc} 1 & 1 & 1 & -2 \\ 0 & 1 & -2/5 & -2 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \\ -5R_3 \\ \\ \end{array} \sim \left[\begin{array}{cccc} 1 & 1 & 1 & -2 \\ 0 & 1 & -2/5 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 + (2/5)R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the homogeneous system

$$\begin{array}{ccccccc} x_1 & & & & & = & 0 \\ & x_2 & & - & 2x_4 & = & 0 \\ & & x_3 & & & = & 0 \end{array}$$

Replacing the variable x_4 with the parameter s, we get

$$\begin{array}{ccccccc} x_1 & & & & & = & 0 \\ & x_2 & & - & 2s & = & 0 \\ & & x_3 & & & = & 0 \end{array}$$

From this, we see that the general solution is

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 2s \\ 0 \\ s \end{array}\right] = s \left[\begin{array}{c} 0 \\ 2 \\ 0 \\ 1 \end{array}\right]$$

B3(d) The coefficient matrix for this system is

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 & -1 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \end{array}\right]$$

To determine the rank and number of parameters, we need to row reduce to row echelon form:

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 & -1 \\ 1 & 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 \end{array}\right] \begin{array}{c} \\ \\ R-3-R_1 \\ R_4 - R-1 \end{array} \sim \left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 0 & -1 \end{array}\right] \begin{array}{c} \\ R_2 \updownarrow R_4 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 2 & 1 & 0 & -1 \end{array}\right] \begin{array}{c} \\ -R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc} 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 2 & 1 & 0 & -1 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \\ R_4 - 2R-2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix} \quad \sim \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

We see that the row echelon form has four pivots, so the rank of the matrix is 4, and the number of parameters in the general solution is 5-4=1. To determine the general solution, I will row reduce our matrix the rest of the way to reduced row echelon form.

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ \\ (-1/2)R_4 \end{matrix} \quad \sim \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_4 \\ R_2 - R_4 \\ R_3 + R_4 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \\ \end{matrix} \quad \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{array}{cccccccc} x_1 & & & + & x_4 & & = & 0 \\ & x_2 & & & & & = & 0 \\ & & x_3 & & & & = & 0 \\ & & & & & x_5 & = & 0 \end{array}$$

Replacing the variable x_4 with the parameter s, we get

$$\begin{array}{cccccccc} x_1 & & & + & s & & = & 0 \\ & x_2 & & & & & = & 0 \\ & & x_3 & & & & = & 0 \\ & & & & & x_5 & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ 0 \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

B4(a) $\left[\begin{array}{ccc|c} 4 & 0 & 6 & 0 \\ 6 & 6 & 3 & -6 \\ -2 & 1 & -4 & -1 \end{array}\right] \begin{array}{c} (1/4)R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 6 & 6 & 3 & -6 \\ -2 & 1 & -4 & -1 \end{array}\right] \begin{array}{c} \\ R_2 - 6R_1 \\ R_3 + 2R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 6 & -6 & -6 \\ 0 & 1 & -1 & -1 \end{array}\right] (1/6)R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This corresponds to the system

$$\begin{array}{rcrcl} x_1 & & + & (3/2)x_3 & = & 0 \\ & x_2 & - & x_3 & = & -1 \end{array}$$

Replacing the variable x_3 with the parameter s, we get

$$\begin{array}{rcrcl} x_1 & & + & (3/2)s & = & 0 \\ & x_2 & - & s & = & -1 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-3/2)s \\ -1+s \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/2 \\ 1 \\ 1 \end{bmatrix}$$

Turning our attention to the related homogeneous system, we would have performed the exact same row operations on the coefficient matrix, leaving us with

$$\begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{array}{rcrcl} x_1 & & + & (3/2)x_3 & = & 0 \\ & x_2 & - & x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter s, we get

$$\begin{array}{rcrcl} x_1 & & + & (3/2)s & = & 0 \\ & x_2 & - & s & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (-3/2)s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -3/2 \\ 1 \\ 1 \end{bmatrix}$$

B4(b) $\left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ -1 & -5 & -5 & -1 \\ -4 & 1 & 9 & 1 \\ -5 & -4 & 0 & 8 \end{array}\right] \begin{array}{c} \\ R_2+R_1 \\ R_3+4R_1 \\ R_4+5R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & -3 & -9 & 9 \\ 0 & 9 & -7 & 41 \\ 0 & 6 & -20 & 58 \end{array}\right] (-1/3)R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & 3 & -3 \\ 0 & 9 & -7 & 41 \\ 0 & 6 & -20 & 58 \end{array}\right] \begin{array}{c} \\ \\ R_3-9R_2 \\ R_4-6R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & -34 & 68 \\ 0 & 0 & -38 & 76 \end{array}\right] \begin{array}{c} \\ \\ (-1/2)R_3 \\ (-1/2)R_4 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4-R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 10 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1+4R_3 \\ R_2-3R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1-2R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system

$$\begin{array}{ccccc} x_1 & & & = & -4 \\ & x_2 & & = & 3 \\ & & x_3 & = & -2 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix}$$

Turning our attention to the related homogeneous system, we would have performed the exact same row operations on the coefficient matrix, leaving us with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{array}{rcl} x_1 \qquad\qquad\qquad & = & 0 \\ x_2 \qquad\quad & = & 0 \\ x_3 & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B4(c) $\left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ -1 & -2 & 5 & -2 & 5 \\ -4 & -1 & 2 & 2 & -4 \\ 5 & 4 & 1 & 8 & 5 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ R_3 + 4R_1 \\ R_4 - 5R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & -3 & 9 & -3 & 9 \\ 0 & -5 & 18 & -2 & 12 \\ 0 & 9 & -19 & 13 & -15 \end{array}\right] \begin{array}{c} \\ (-1/3)R_2 \\ \\ \\ \end{array}$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & 1 & -3 & 1 & -3 \\ 0 & -5 & 18 & -2 & 12 \\ 0 & 9 & -19 & 13 & -15 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 5R_2 \\ R_4 - 9R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & 1 & -3 & 1 & -3 \\ 0 & 0 & 3 & 3 & -3 \\ 0 & 0 & 8 & 4 & 12 \end{array}\right] \begin{array}{c} \\ \\ (1/3)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & 1 & -3 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 8 & 4 & 12 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 8R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & 1 & -3 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -4 & 20 \end{array}\right] \begin{array}{c} \\ \\ \\ (-1/4)R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ 0 & 1 & -3 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] \begin{array}{c} R_1 + R_4 \\ R_2 - R_4 \\ R_3 - R_4 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 4 & 0 & -1 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] \begin{array}{c} R_1 - 4R_3 \\ R_2 + 3R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -17 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$$

This corresponds to the system

$$\begin{array}{rcl} x_1 \qquad\qquad\qquad\qquad & = & -3 \\ x_2 \qquad\qquad\quad & = & 14 \\ x_3 \qquad\quad & = & 4 \\ x_4 & = & -5 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 14 \\ 4 \\ -5 \end{bmatrix}$$

Turning our attention to the related homogeneous system, we would have performed the exact same row operations on the coefficient matrix, leaving us with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This corresponds to the system

$$\begin{array}{ccccccccc} x_1 & & & & & & & = & 0 \\ & & x_2 & & & & & = & 0 \\ & & & & x_3 & & & = & 0 \\ & & & & & & x_4 & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

B4(d) $$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 4 & 4 & 6 & -8 & 4 & -4 \\ 1 & 1 & 4 & -2 & 1 & -6 \\ 3 & 3 & 2 & -4 & 5 & 6 \end{array}\right] \begin{array}{c} \\ R_2 - 4R_1 \\ R_2 - R_1 \\ R_4 - 3R_1 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 0 & 0 & -6 & -12 & -12 & -12 \\ 0 & 0 & 1 & -3 & -3 & -8 \\ 0 & 0 & -7 & -7 & -7 & 0 \end{array}\right] \begin{array}{c} \\ (-1/6)R_2 \\ \\ (-1/7)R_4 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & -3 & -3 & -8 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & -5 & -5 & -10 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{array}\right] \begin{array}{c} \\ \\ (-1/5)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 3R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to the system

$$\begin{array}{ccccccccccc} x_1 & + & x_2 & & & & & + & 3x_5 & = & 6 \\ & & & & x_3 & & & & & = & -2 \\ & & & & & & x_4 & + & x_5 & = & 2 \end{array}$$

Replacing the variable x_2 with the parameter s and the variable x_5 with the parameter t, we get

$$\begin{array}{rcrcrcrcrcr} x_1 & + & s & & & & & + & 3t & = & 6 \\ & & & & x_3 & & & & & = & -2 \\ & & & & & & x_4 & + & t & = & 2 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-s-3t \\ s \\ -2 \\ 2-t \\ t \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Turning our attention to the related homogeneous system, we would have performed the exact same row operations on the coefficient matrix, leaving us with

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & & & & & + & 3x_5 & = & 0 \\ & & & & x_3 & & & & & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \end{array}$$

Replacing the variable x_2 with the parameter s and the variable x_5 with the parameter t, we get

$$\begin{array}{rcrcrcrcrcr} x_1 & + & s & & & & & + & 3t & = & 0 \\ & & & & x_3 & & & & & = & 0 \\ & & & & & & x_4 & + & t & = & 0 \end{array}$$

From this, we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-3t \\ s \\ 0 \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Instructor's comments As you worked through problem B4, you hopefully noticed the following shortcut: to go from the general solution of the non-homogeneous system to the general solution of the homogeneous solution, you simply need to switch the "constant vector" (that is, the vector not affiliated with any parameters) to the zero vector. In fact, the constant vector comes from the augmented column, which in the case of the homogeneous system is all zeros.

D1(a) If $\vec{x}$ is orthogonal to $\vec{a}$, then $\vec{x} \cdot \vec{a} = 0$. That is, we must have that $a_1x_1 + a_2x_2 + a_3x_3 = 0$. Similarly, if $\vec{x}$ is orthogonal to $\vec{b}$, then $b_1x_1 + b_2x_2 + b_3x_3 = 0$, and if $\vec{x}$ is orthogonal to $\vec{c}$, then $c_1x_1 + c_2x_2 + c_3x_3 = 0$. So, in order for $\vec{x} \neq \vec{0}$ to be simultaneously orthogonal to $\vec{a}$, $\vec{b}$, and $\vec{c}$, $\vec{x}$ must be a non-trivial solution to the following system of homogeneous equations:

$$\begin{array}{ccccccc} a_1x_1 & + & a_2x_2 & + & a_3x_3 & = & 0 \\ b_1x_1 & + & b_2x_2 & + & b_3x_3 & = & 0 \\ c_1x_1 & + & c_2x_2 & + & c_3x_3 & = & 0 \end{array}$$

D1(b) The matrix A is the coefficient matrix for the system of homogeneous equations given in part (a). Since all homogeneous systems are consistent, we know that we can apply part 2 of Theorem 2. Since the system will have non-trivial solutions if and only if there is at least one parameter in the general solution, then we know that in order for there to be a non-trivial solution to the system from (a), the rank of A must be less than 3.