Lecture 2m

Bases

(pages 97-99)

After looking again at spanning sets and linear independence, it makes sense to take another look at their combination–a basis. First, let's recall our definition from Lecture 1i:

Definition: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a spanning set for a subspace S of $\mathbb{R}^n$ and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is called a **basis** for S.

Recalling further that $\mathbb{R}^n$ is itself a subspace of $\mathbb{R}^n$, lets focus briefly on what a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ would need to look like to be a basis for $\mathbb{R}^n$. First, we know that the set must span all of $\mathbb{R}^n$, and so by Theorem 2, we know that $k \geq n$. But we also know that the set must be linearly independent, and thus by Theorem 4 we know that $k \leq n$. The only way to have both of these happen at the same time is if $k = n$. And so we see that any basis for $\mathbb{R}^n$ must have exactly n vectors in it! But of course, not every set of n vectors is a basis for $\mathbb{R}^n$. The condition that they are linearly independent means that the coefficient matrix of $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{0}$ must have rank n. But imagine what this would mean for the question of span. Because in order to span all of $\mathbb{R}^n$, we need the equation $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{v}$ to have a solution for any choice of $\vec{v}$. That is, we need to know that there is no choice of $\vec{v}$ that could lead to a bad row. But the only way for that to happen is if there is a pivot in every row of the RREF of the coefficient matrix. That is, we need to know that the rank of the coefficient matrix is n. Oh, but we already have that! So we see that once we've got the proper number of vectors, the condition to be linearly independent is the same as the condition to span all of $\mathbb{R}^n$, namely that the rank of the coefficient matrix is also n. And this is the "REALLY BIG" result I've been mentioning:

Theorem 5: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{v}$ is n. (Where $\vec{v}$ is an arbitrary vector from $\mathbb{R}^n$–since we're only looking at the coefficient matrix, it doesn't matter what $\vec{v}$ is.)

With that in mind, let's look back over our examples from the previous two lectures:

Example: Our first example in lecture 2k had us write $\begin{bmatrix} -1 \\ 4 \\ 8 \\ 8 \end{bmatrix}$ as a linear combination of vectors in $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix} \right\}$. During the process, we row reduced the relevant coefficient matrix (it was augmented, but we can just ignore the augmented column), and saw that the rank of the coefficient matrix was 4. So we see that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Example: The next two examples in lecture 2k had us looking at

$$A = \left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$$

The first example has us show that $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ was in the span of this set, and looking at the process now, we see that the rank of the coefficient matrix was 2. At the time, this corresponded to the fact that there was only one way to write $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in A. Now, we use the fact that $2 \leq 3$ to see that A is linearly independent. But we know that A can not be a basis for $\mathbb{R}^3$, because it does not have the correct number of vectors, and in fact the third example in lecture 2.11 showed that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ was not in Span$A$, thus providing a specific example of the fact that Span$A \neq \mathbb{R}^3$.

Example: The last example in lecture 2k had us write $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the vectors in $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. During this process, we see that the rank of the coefficient matrix is 2. So, this is an example of a set that has the correct number of vectors, but the wrong rank. This means that the set is both not linearly independent and also does not span all of $\mathbb{R}^3$.

Example: In the first example of lecture 2l, we saw that the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. We did this by showing that the rank of the coefficient matrix is 3, and thus we now know that $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Example: In the second example of lecture 2l, we saw that the set

$$\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$$

is linearly dependent. We did this by showing that the rank of the coefficient matrix is 2, and thus we now know that $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^3$. Well, actually we knew that as soon as we knew that it wasn't linearly independent. But it's always nice to see another example where the right number of vectors does not necessarily mean that we have a basis.

Example: In the last example of lecture 2l, we determined that the set

$$\left\{ \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 13 \\ -4 \end{bmatrix}, \begin{bmatrix} -16 \\ 3 \\ 8 \end{bmatrix} \right\}$$

could not be linearly independent because it has too many vectors. Again, as soon as we know that it isn't linearly independent, we know that it isn't a basis, but we can also immediately rule it out as a possible basis for $\mathbb{R}^3$ because it doesn't have the right number of vectors.

Well, we've pretty much covered every possibility of being a basis for $\mathbb{R}^3$, and hopefully you can extrapolate to any $\mathbb{R}^n$. So let's turn our attention now to determining whether a set of vectors $\mathcal{B}$ can be a basis for a subspace S of some $\mathbb{R}^n$. The first thing to realize is that linear independence does not depend on any particular subspace, so we still know that there can be at most n vectors in any linearly independent set, and that we'll know that a set of k vectors is linearly independent if and only if the rank of the coefficient matrix is k. But how many vectors do we need to span S? Well, this depends on the subspace S itself. But there are some results we can prove about the number of vectors in the basis for a subspace, similar to our results for $\mathbb{R}^n$. We'll start with

<u>Lemma 2.3.6</u>: Suppose that S is a non-trivial subspace of $\mathbb{R}^n$ and Span $\{\vec{v}_1, \ldots, \vec{v}_l\} = S$. If $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is a linearly independent set of vectors in S, then $k \leq l$.

What does this Lemma mean? Well, we start out with a spanning set for S ($\{\vec{v}_1, \ldots, \vec{v}_l\}$), but we don't know if it is linearly independent. So then we look at another subset of S ($\{\vec{u}_1, \ldots, \vec{u}_k\}$) that we know is linearly independent, but may not span all of S. The point of this Lemma is that, even though we potentially have a situation where neither set is a basis for S, we want to notice the number of vectors in our spanning set is greater than or equal to the number of vectors in our linearly independent set. This makes sense from our intuition about spanning sets versus linearly independent sets–spanning sets need to have lots of vectors to make sure that they span all of S, while linearly independent sets can't have too many vectors or else they start repeating themselves. The thing we really want to notice about the lemma is that it has changed the maximum number of vectors we can have in our basis for S. Before, we were limited only by the fact that we knew a linearly independent set of vectors from $\mathbb{R}^n$ could have at most n vectors. But now, IF we find any spanning set of S that has l vectors, where l is quite likely to be less than n, then the new maximum number of VECTORS FROM S that can be linearly independent is l.

The proof of the Lemma is given in the book, so I won't concern myself with it here. And the Lemma itself is only useful if you happen to have a spanning set for S already. So, as was the case with our lemma for spanning sets and our Lemma for linear independence, the value in this lemma is what we can do with it:

<u>Theorem 7</u>: If $\{\vec{v}_1, \ldots, \vec{v}_l\}$ and $\{\vec{u}_1, \ldots, \vec{u}_k\}$ are both bases of a non-trivial subspace S of $\mathbb{R}^n$, then $k = l$.

So, in Theorem 2.3.5 we learned that every basis for $\mathbb{R}^n$ will have exactly n vectors in it, and now Theorem 2.3.7 tells us that every basis for a subspace S of $\mathbb{R}^n$ will have exactly k vectors in it, where what the number k is depends on the S. This magic number is so fabulous, we actually give it a name:

<u>Definition</u>: If S is a non-trivial subspace of $\mathbb{R}^n$ with a basis containing k vectors, then we say that the **dimension** of S is k and write $\dim S = k$.

<u>Note</u>: We will choose to say that the empty set is a basis for the subspace $\left\{\vec{0}\right\}$ of $\mathbb{R}^n$, and thus that the dimension of $\left\{\vec{0}\right\}$ is 0.

Of course, Theorem 2.3.7 doesn't actually help you find a basis for a S. What the basis vectors turn out to be will depend on what S is, so we will need some sort of description of S before we can determine whether or not a set of vectors is a basis for it.

Example: Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the hyperplane $x_1 + 2x_2 + x_3 + 2x_4 = 0$.

To show that $\mathcal{B}$ is a basis for the hyperplane, we need to see that it is both linearly independent, and that it spans the hyperplane. To show that $\mathcal{B}$ is linearly independent, we need to row reduce the coefficient matrix of the system

$$t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$$

$\mathcal{B}$ is linearly independent if and only if the rank of the coefficient matrix is 3. But before we rush ahead and answer this question, let's consider the question of span. In order to show that $\mathcal{B}$ spans the hyperplane, we need to see that every vector in the hyperplane can be written as a linear combination of the vectors in $\mathcal{B}$. That is to say, we need to see that the system

$$t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

has a solution for every possible choice of $x_2, x_3, x_4 \in \mathbb{R}$. We will do this by row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2x_2 - x_3 - 2x_4 \\ 0 & 1 & -1 & x_2 \\ -1 & 0 & 1 & x_3 \\ 0 & -1 & 0 & x_4 \end{array}\right]$$

and to show that the row echelon form does not have a bad row. Since the coefficient matrix for this system is the same as the coefficient matrix for the homogeneous system we need to look at to determine linear independence, we will be able to show that $\mathcal{B}$ is linearly independent at the same time that we show it spans the hyperplane. Let's get started!

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2x_2 - x_3 - 2x_4 \\ 0 & 1 & -1 & x_2 \\ -1 & 0 & 1 & x_3 \\ 0 & -1 & 0 & x_4 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2x_2 - x_3 - 2x_4 \\ 0 & 1 & -1 & x_2 \\ 0 & 0 & 2 & -2x_2 - 2x_4 \\ 0 & -1 & 0 & x_4 \end{array}\right] \begin{array}{c} \\ \\ (1/2)R_3 \\ R_4 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2x_2 - x_3 - 2x_4 \\ 0 & 1 & -1 & x_2 \\ 0 & 0 & 1 & -x_2 - x_4 \\ 0 & 0 & -1 & x_2 + x_4 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2x_2 - x_3 - 2x_4 \\ 0 & 1 & -1 & x_2 \\ 0 & 0 & 1 & -x_2 - x_4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We now see that the rank of the coefficient matrix is 3, so $\mathcal{B}$ is linearly independent, and that there is no way for the row echelon form to have a bad row, so $\mathcal{B}$ spans the hyperplane. Thus, we have shown that $\mathcal{B}$ is a basis for the hyperplane $x_1 + 2x_2 + x_3 + 2x_4 = 0$. And thus we have also shown that the dimension of the hyperplane is 3.

Example: We know that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ can not be a basis for the plane $x_1 + 2x_2 + x_3 = 0$, because a plane is the span of two linearly independent vectors, and thus every plane has dimension 2. Because $\mathcal{B}$ does not have 2 vectors, it cannot be a basis for *any* plane.