Lecture 2l

Linear Independence Problems

(pages 95- 96)

In the last lecture, we were looking at linear combinations of vectors, and whether or not there was a way to write a vector $\vec{v}$ as a linear combination of the vectors in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$. But there is one vector that can always be written as a linear combination of any (non-empty) set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$: the zero vector. And so, one does not ask whether or not $\vec{0}$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, but rather how many ways can $\vec{0}$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$. On that matter, recall the following definition from Lecture 1i:

Definition: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be **linearly independent** if the only solution to

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is $t_1 = \cdots = t_k = 0$.

At this point, we can now see that the vector equation $\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ is actually a homogeneous system of equations, and as such we can determine whether or not a system is linearly independent by row reducing the coefficient matrix.

Example: Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to see if there are any parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & - & 3t_3 & = & 0 \\ 2t_1 & + & 4t_2 & - & 4t_3 & = & 0 \\ -t_1 & + & 7t_2 & & & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 4 & -4 \\ -1 & 7 & 0 \end{bmatrix} \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ 0 & 8 & -3 \end{bmatrix} \; (1/2)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 8 & -3 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - 8R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables (which is the same as the number of vectors), there are no parameters in the general solution. And this means that the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ IS linearly independent.

Example: Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Again, this will come down to determining whether or not there are parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} & & t_2 & + & 3t_3 & = & 0 \\ 3t_1 & - & 7t_2 & & & = & 0 \\ -2t_1 & + & 6t_2 & + & 4t_3 & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ -2 & 6 & 4 \end{bmatrix} (-1/2)R_3 \sim \begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ 1 & -3 & -2 \end{bmatrix} R_1 \updownarrow R_3$$
$$\sim \begin{bmatrix} 1 & -3 & -2 \\ 3 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} R_2 - 3R_1 \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} R_3 - (1/2)R_2$$
$$\sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 2. This means that there is one parameter in the general solution to the homogeneous system. And this means that the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ in NOT linearly independent. (That is, our set is linearly dependent.)

From these examples, we see that the rank of the coefficient matrix will determine whether or not a set of vectors is linearly independent. We summarize our observations in the following:

<u>Lemma 2.3.3</u>: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly independent if and only if the rank of the coefficient matrix of the homogeneous system $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$ is k.

As in the previous lecture, though this may seem like the big result, the one we'll use for the REALLY BIG result is the following:

Theorem 2.3.4: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

In general, we actually use Theorem 2.3.4 to determine that a set of vectors is linearly DEPENDENT:

Example: The set of vectors $\left\{ \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 13 \\ -4 \end{bmatrix}, \begin{bmatrix} -16 \\ 3 \\ 8 \end{bmatrix} \right\}$ is linearly dependent, since the set contains four vectors, and by theorem 4, only sets with three or fewer vectors from $\mathbb{R}^3$ can be linearly independent.