Lecture 2k

Spanning Problems

(pages 91-95)

Systems of linear equations came up frequently in our study of $\mathbb{R}^n$. Because we had not developed the techniques we now have for solving systems of linear equations, we were unable to fully explore $\mathbb{R}^n$ in Chapter 1. So now, we will go back and revisit some of our definitions from Chapter 1, and work on finding solutions to common questions. To that end, consider the following example:

Example: Write $\begin{bmatrix} -1 \\ 4 \\ 8 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix}$. That is, find scalars $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 8 \\ 8 \end{bmatrix}$$

This single vector equation is solved by breaking it into its four components, giving us the following system of linear equations:

$$\begin{array}{rcrcrcrcr} t_1 & + & 2t_2 & + & t_3 & & & = & -1 \\ & & 2t_2 & + & 8t_3 & + & 4t_4 & = & 4 \\ t_1 & - & 2t_2 & + & 5t_3 & + & 3t_4 & = & 8 \\ & - & 3t_2 & - & 5t_3 & - & t_4 & = & 8 \end{array}$$

And we solve this system of linear equations by row reducing its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 2 & 8 & 4 & 4 \\ 1 & -2 & 5 & 3 & 8 \\ 0 & -3 & -5 & -1 & 8 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_1 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 2 & 8 & 4 & 4 \\ 0 & -4 & 4 & 3 & 9 \\ 0 & -3 & -5 & -1 & 8 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 2 & 2 \\ 0 & -4 & 4 & 3 & 9 \\ 0 & -3 & -5 & -1 & 8 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 2 & 2 \\ 0 & 0 & 20 & 11 & 17 \\ 0 & 0 & 7 & 5 & 14 \end{array}\right] \begin{array}{c} \\ \\ (1/20)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 2 & 2 \\ 0 & 0 & 1 & 11/20 & 17/20 \\ 0 & 0 & 7 & 5 & 14 \end{array}\right] R_4 - 7R_3 \quad \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 2 & 2 \\ 0 & 0 & 1 & 11/20 & 17/20 \\ 0 & 0 & 0 & 23/20 & 161/20 \end{array}\right] (20/23)R_4$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 2 & 2 \\ 0 & 0 & 1 & 11/20 & 17/20 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{c} R_2 - 2R_4 \\ R_3 - (11/20)R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & 0 & -12 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ R_2 - 4R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right] R_1 - 2R_2 \quad \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

From the final RREF matrix, we see that the system has the unique solution $t_1 = 2$, $t_2 = 0$, $t_3 = -3$, and $t_4 = 7$. This means that we can write $\begin{bmatrix} -1 \\ 4 \\ 8 \\ 8 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix}$ as follows:

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \\ -2 \\ -3 \end{bmatrix} + -3\begin{bmatrix} 1 \\ 8 \\ 5 \\ -5 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 8 \\ 8 \end{bmatrix}$$

Back in Lecture 1h, we defined the span of a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ to be the set of all linear combinations of these vectors. As we just saw in the previous example, we can use a system of linear equations to find the scalars used to write one vector as the linear combination of a set of vectors. But what if our system did not have a solution? That is, what if the system was inconsistent? This would correspond to the case where our vector was not in the span of the set of vectors. So, if we know that a vector $\vec{v}$ is in Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then we can set up a system of linear equations whose solution provides us with the scalars that show $\vec{v}$ is a linear combination $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. But what we also get, from this same system, is the fact that a vector $\vec{v}$ is in Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ if and only if the corresponding system of equations is consistent.

Example: Determine whether or not $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is in Span$\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

To solve this we need to determine whether or not there are any solutions to the vector equation $t_1 \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} + t_2 \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Breaking this vector equation into its three components, we see that the vector equation has a solution if and only if the following system of linear equations has a solution:

$$\begin{array}{rcrcl} -3t_1 & + & 2t_2 & = & 1 \\ 2t_1 & - & 4t_2 & = & 2 \\ 6t_3 & - & 13t_2 & = & 4 \end{array}$$

To determine whether or not this system has a solution, we will write it as an augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} -3 & 8 & 1 \\ 2 & -4 & 2 \\ 6 & -13 & 4 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ \\ \end{array} \sim \left[\begin{array}{cc|c} -3 & 8 & 1 \\ 1 & -2 & 1 \\ 6 & -13 & 4 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ -3 & 8 & 1 \\ 6 & -13 & 4 \end{array}\right] \begin{array}{c} \\ R_2 + 3R_1 \\ R_3 - 6R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The final matrix is in row echelon form, and as it has no bad rows, we know that the system is consistent. This means that $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ IS in Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

Example: Determine whether or not $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

To solve this we need to determine whether or not there are any solutions to the vector equation $t_1 \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} + t_2 \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Breaking this vector equation into its three components, we see that the vector equation has a solution if and only if the following system of linear equations has a solution:

$$\begin{array}{rcrcl} -3t_1 & + & 2t_2 & = & 1 \\ 2t_1 & - & 4t_2 & = & 2 \\ 6t_3 & - & 13t_2 & = & 3 \end{array}$$

To determine whether or not this system has a solution, we will write it as an augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} -3 & 8 & 1 \\ 2 & -4 & 2 \\ 6 & -13 & 3 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ \\ \end{array} \sim \left[\begin{array}{cc|c} -3 & 8 & 1 \\ 1 & -2 & 1 \\ 6 & -13 & 3 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ -3 & 8 & 1 \\ 6 & -13 & 3 \end{array}\right] \begin{array}{c} \\ R_2 + 3R_1 \\ R_3 - 6R_1 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{c} \\ (1/2)R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -3 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

Since the bottom row in the last matrix is a bad row, we know that the system is inconsistent, and therefore $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is NOT in Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} \right\}$.

And so, to summarize, to determine if a vector $\vec{v}$ is in Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, we row reduce the augmented matrix $[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n|\vec{v}]$. If a row echelon form has a bad row, then $\vec{v}$ is not in the span. If the row echelon form does not have a bad row, then $\vec{v}$ is in the span. In the case that $\vec{v}$ is in the span, any solution to the system gives us the coefficients needed to write $\vec{v}$ as a linear combination of the vectors in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. And yes, it is possible to have multiple solutions:

Example: Write $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the vectors in $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

We need to find scalars $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. To that end, we will row reduce the affiliated augmented matrix as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ -R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Putting our RREF matrix back into equations, we see that

$$\begin{array}{ccccccc} t_1 & & & + & t_3 & = & 3 \\ & & t_2 & - & t_3 & = & -1 \end{array}$$

Replacing the variable t_3 with the scalar s, we get

$$\begin{array}{ccccccc} t_1 & & & + & s & = & 3 \\ & & t_2 & - & s & = & -1 \end{array}$$

From this we see that the general solution to our system is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 3-s \\ -1+s \\ s \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

And from this general solution, we see that (using $s = 0$)

$$3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

and that (using $s = 1$)

$$2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

and that (using $s = 3$)

$$0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

and so on!

We'll see in the next lecture that we only get multiple ways to write a vector $\vec{v}$ as a linear combination of the vectors in $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ when the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent. But before we leave the subject of spanning sets behind, there is one other question to consider: When does a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ span all of $\mathbb{R}^n$? That is equivalent to asking whether or not the equation $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{v}$ has a solution for every $\vec{v} \in \mathbb{R}^n$. In terms of our technique for solving such systems, this means that no matter what the augmented column is, our augmented matrix can NOT have a bad row. The only way to achieve this is to make sure that every row in the COEFFICIENT matrix has a pivot. That is, we need the rank of the coefficient matrix to equal the number of equations, which is n. So, we've proved the following result:

Lemma 2.3.1: A set of k vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ spans all of $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of the system $t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k = \vec{v}$ is n.

Now, you would think that this would be a "Theorem", but in fact the following result will turn out to be the important one in the end:

Theorem 2.3.2: Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$. If $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = \mathbb{R}^n$, then $k \geq n$.

We'll use Theorem 2.3.2 later to prove the REALLY BIG RESULT!!! But for now, let's get some practice with spanning sets...