Lecture 2j

Homogeneous Linear Equations

(pages 86-87)

Definition: A linear equation is **homogeneous** (pronounced ho-mo-gene-ee-ous) if the right-hand side is zero. A system of linear equations is homogeneous if all of the equations of the system are homogeneous.

Now, homogeneous systems are still systems of linear equations, so we can solve them using our established techniques.

Example: Find the general solution of the homogeneous system

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 0 \end{array}$$

Now, before we even put this into matrix form, there is one solution that is immediately obvious: $\vec{x} = \vec{0}$. In fact, $\vec{0}$ will be a solution to every homogeneous system, and it is so obvious a solution that it is called the **trivial solution**. But, this means that when we are looking at a homogeneous system, we do not need to ask whether or not the system is consistent, as every homogeneous system is consistent. But, one solution is not necessarily the whole solution, so let's go ahead and turn our system into an augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] (1/4)R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 6R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Turning our RREF matrix back into equations, we have

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

We need to replace the variable x_2 with the parameter s, giving us

$$\begin{array}{rcrcrcl} x_1 & + & 2s & & & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

From this we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

But the thing we want to learn from this example is not what the general solution is, but rather if we take a look at the matrices generated by our row reduction steps, we see that the last column was *always* a column of zeros. This is because row operations happen within a column, and it doesn't matter if we switch a zero for a zero, multiply zero by a non-zero scalar, or add a multiple of zero to zero–we always end up with zero! For that reason, people usually drop the augmented column, and focus only on the coefficient matrix, when they are dealing with a homogeneous system.

Example: Find the general solution of the homogeneous system

$$\begin{array}{rcrcl} x_1 & + & 7x_2 & = & 0 \\ -3x_1 & - & 3x_2 & = & 0 \end{array}$$

To solve this, we will row reduce the COEFFICIENT matrix, as follows:

$$\begin{bmatrix} 1 & 7 \\ -3 & -3 \end{bmatrix} \; R_2 + 3R_1 \; \sim \begin{bmatrix} 1 & 7 \\ 0 & 18 \end{bmatrix} \; (1/18)R_2 \; \sim \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \; R_1 - 7R_2$$
$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Turning our RREF matrix back into equations, we have

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \end{array}$$

From this we see that the only solution is $\vec{x} = \vec{0}$.