Lecture 2i

The Rank of a Matrix

(pages 85-86)

Okay folks, you are about to experience the easiest lecture of the term. And it all starts with this:

Definition: The **rank** of a matrix A is the number of leading 1s in its reduced row echelon form, and is denoted by $\text{rank}(A)$.

Example: Find the rank of $\begin{bmatrix} 1 & -3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Well, our matrix is already in reduced row echelon form, so we immediately see that it has three leading 1s, so the rank is 3.

See, it's that easy! Basically, the term "rank" is just giving a name to a quantity that we have already been concerning ourselves with.

Example: Find the rank of $\left[\begin{array}{ccc|c} 2 & -2 & 6 & 4 \\ -3 & -1 & 11 & 10 \\ 2 & -4 & 4 & 6 \end{array}\right]$.

First, we will need to put the matrix into reduced row echelon form:

$$\left[\begin{array}{ccc|c} 2 & -2 & 6 & 4 \\ -3 & -1 & 11 & 10 \\ 2 & -4 & 4 & 6 \end{array}\right] \begin{array}{c} (1/2)R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ -3 & -1 & 11 & 10 \\ 2 & -4 & 4 & 6 \end{array}\right] \begin{array}{c} \\ R_2 + 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & -2 & -2 & -2 \end{array}\right] (1/2)R_2 \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -2 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] \begin{array}{c} \\ \\ (1/2)R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 - 2R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 + R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

This last matrix is in RREF, and we see that it has three leading 1s. So the rank of our matrix is 3.

In fact, all of the matrices in this example have a rank of 3, since they are all row equivalent to our final RREF matrix. There are a couple of other things we can learn from this example. The first is to emphasize the fact that RREF and rank are a property of a matrix, not a system of linear equations. Since the "|" in our augmented matrices is not a property of a matrix, but instead is simply something we add to separate the coefficient matrix from the right side of the equations, do not let the "|" distract you from the form of a RREF matrix. I often find that students forget that there can be a pivot/leading 1 in the augmented column. Now, this only happens when the corresponding system is inconsistent. And this is a perfect lead-in to our next theorem:

Theorem 2.2.2: Let $[A|\vec{b}]$ be a system of m linear equations in n variables.

(1) The system is consistent if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|\vec{b}]$.

(2) If the system is consistent, then the number of parameters in the general solution is the number of variables minus the rank of the matrix:

$$\#\text{ of paramters} = n - \text{rank}(A)$$

This theorem is essentially the same as Theorem 2.1.1, rewritten to make us of our new terminology. It does look nicer, doesn't it?

Before we move on, there is one last thing to note about the rank of a matrix. You do not actually need to row reduce all the way to reduced row echelon form to determine the rank. Because RREF is unique, it is convenient to use it when making definitions or writing theorems, but the number, and even location, of the leading 1s of the RREF can be determined from any row echelon form, since the leading 1s of RREF appear in the same place as the pivots in row echelon form. (In fact, some textbooks require that the pivot in row echelon form be a 1, making the only difference between REF and RREF the existence of non-zero terms above the pivot.) So, back in my example, as soon as I got to the matrix $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right]$, I saw that there were three pivots in this row echelon form, and therefore knew that the rank of the matrix was 3.