Lecture 2h

Reduced Row Echelon Form

(pages 83-85)

So, row echelon form is great, right? Want to know if a system has a solution? Row echelon form can tell you that! It will even tell you how many parameters are in that solution. The only place where row echelon form falls short is that it doesn't actually tell you the solution. Now, all it takes are some back-substitution steps, and viola, you've got the solution. But it turns out that instead of doing back-substitution, we can instead do a few more row operations, and we end up with a matrix from which the general solution is immediately clear. Such a matrix is said to be in *reduced* row echelon form.

Definition: A matrix R is said to be in **reduced row echelon form (RREF)** if

(1) It is in row echelon form.
(2) All leading entries are 1, called a **leading 1**.
(3) In a column with a leading 1, all the other entries are zeros.

As we see from property (1), every matrix in reduced row echelon form is also in row echelon form. This means that we don't need to change the procedure we established in the last section for finding the general solution. The process of substituting in parameters and turning things back into equations to solve for our variables is still what we are going to do–it will simply be much easier!

Example: Consider the following augmented matrix for a system of linear equations, already in row echelon form:

$$\left[\begin{array}{cc|c} 1 & -3 & -5 \\ 0 & 1 & 4 \end{array}\right]$$

Instead of switching back to equations and using back-substitution, consider instead performing one more row operation: $R_1 + 3R_2$. This would give us the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

If we now switch back to equations, we have the system

$$\begin{aligned} x_1 &= 7 \\ x_2 &= 4 \end{aligned}$$

No back-substitution is necessary–we can simply read off the solution! Let's look at a larger example:

Example: Consider the following augmented matrix for a system of linear equations, already in row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 8 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Instead of doing back-substitution, we will instead perform more elementary row operations, with the intent to make the entries ABOVE our leading 1s into zeros:

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 8 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 5 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \; R_1 - 5R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

As in the previous example, we can now quickly read off the general solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \\ 3 \end{bmatrix}$$

Example: Find the general solution to the system of linear equations that is equivalent to the following augmented matrix in reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 9 & 0 & -5 \\ 0 & 1 & 2 & -5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Well, let's turn the rows back into equations:

$$\begin{array}{ccccccccc} x_1 & & & & & + & 9x_4 & & = & -5 \\ & & x_2 & + & 2x_3 & - & 5x_4 & & = & 7 \\ & & & & & & & x_5 & = & 3 \end{array}$$

We see that the leading variables are x_1, x_2 and x_5, so we replace the variable x_3 with the parameter s, and the variable x_4 with the parameter t. This turns our system into

$$\begin{array}{ccccccccc} x_1 & & & & & + & 9t & & = & -5 \\ & & x_2 & + & 2s & - & 5t & & = & 7 \\ & & & & & & & x_5 & = & 3 \end{array}$$

The third equation tells us that $x_5 = 3$. The second equation tells us that $x_2 = 7 - 2s + 5t$. And the first equation tells us that $x_1 = -5 - 9t$. But notice that I did not need to substitute the information from equation three into equation two to solve for x_2, nor did I need to substitute the information from equation three or two to solve for x_1 in equation 1. So we no longer have any back-substitution steps! And we see that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 9t \\ 7 - 2s + 5t \\ s \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

In addition to being the closest thing to a solution that a matrix is ever going to be, reduced row echelon form has one other excellent property:

<u>Theorem 2.2.1</u>: For any given matrix A there is a unique matrix in reduced row echelon form that is row equivalent to A.

Thanks to this Theorem, we can now refer to THE reduced row echelon form of a matrix, meaning the unique matrix in reduced row echelon form that it is row equivalent to. So, unlike with row echelon form, where I had to always say "your solutions may be different," when we reduce all the way to RREF we will end up with the same matrix, even if we use different elementary row operations. The proof of this theorem is outside the scope of this course, but we will nevertheless make use of this fact from time to time.

One last thing before you embark on row reducing matrices into reduced row echelon form. The textbook comments on the possible algorithms for placing a matrix in *reduced* row echelon form. The algorithm that involves the least calculations is to first put a matrix into row echelon form (during which you will eliminate the non-zero terms BELOW each pivot), and then to proceed to eliminate the non-zero terms ABOVE each pivot IN REVERSE ORDER. That is, if row 7 was the last row to have a pivot, then you will first eliminate the non-zero terms above the pivot in row 7, and then the non-zero terms above the pivot in row 6, and then the non-zero terms above the pivot in row 5, and so on. So why don't we just eliminate the non-zero terms above and below a pivot at the same time? Well, as I said, that actually turns out to be more calculations, but it is true that it turns out to be fewer Steps (in the sense of the algorithm I gave to get a matrix into row echelon form), and it also means writing fewer matrices. Personally, I will always give solutions using the method of first getting into row echelon form and then proceeding to reduced row echelon form, but if you want to go straight to reduced row echelon form that's just fine with me.