Solution to Practice 2g

B7 Let a be the price of an armchair, s be the price of a sofa bed, and d be the price of a double bed. Then to solve this problem, we need to find the general solution to the following system of linear equations

$$\begin{array}{ccccccc} 20a & + & 10s & + & 8d & = & 15200 \\ 15a & + & 12s & + & 10d & = & 15700 \\ 12a & + & 20s & + & 10b & = & 19600 \end{array}$$

To do this, lets look at the augmented matrix for this system:

$$\left[\begin{array}{ccc|c} 20 & 10 & 8 & 15200 \\ 15 & 12 & 10 & 15700 \\ 12 & 20 & 10 & 19600 \end{array}\right]$$

We then find a row equivalent matrix in row echelon form as follows:

$$\left[\begin{array}{ccc|c} 20 & 10 & 8 & 15200 \\ 15 & 12 & 10 & 15700 \\ 12 & 20 & 10 & 19600 \end{array}\right] \begin{array}{c} (1/20)R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 2/5 & 760 \\ 15 & 12 & 10 & 15700 \\ 12 & 20 & 10 & 19600 \end{array}\right] \begin{array}{c} \\ R_2 - 15R_1 \\ R_3 - 12R_1 \end{array}$$

$$\left(\begin{array}{cccc} -15(1 & 1/2 & 2/5 & 760) \\ \hline -15 & -15/2 & -6 & -11400 \\ 15 & 12 & 10 & 15700 \\ \hline 0 & 9/2 & 4 & 4300 \end{array}\right) \left(\begin{array}{cccc} -12(1 & 1/2 & 2/5 & 760) \\ \hline -12 & -6 & -24/5 & -9120 \\ 12 & 20 & 10 & 19600 \\ \hline 0 & 14 & 26/5 & 10480 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & 2/5 & 760 \\ 0 & 9/2 & 4 & 4300 \\ 0 & 14 & 26/5 & 10480 \end{array}\right] (2/9)R_2 \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 2/5 & 760 \\ 0 & 1 & 8/9 & 8600/9 \\ 0 & 14 & 26/5 & 10480 \end{array}\right] R_3 - 14R_2$$

$$\left(\begin{array}{cccc} -14(0 & 1 & 8/9 & 8600/9) \\ \hline 0 & -14 & -112/9 & -120400/9 \\ 0 & 14 & 26/5 & 10480 \\ \hline 0 & 0 & -326/45 & -26080/9 \end{array}\right) \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 2/5 & 760 \\ 0 & 1 & 8/9 & 8600/9 \\ 0 & 0 & -326/45 & -26080/9 \end{array}\right]$$

Putting the row echelon form matrix back into equations, we get

$$\begin{array}{ccccccc} a & + & (1/2)s & + & (2/5)d & = & 760 \\ & & s & + & (8/9)d & = & 8600/9 \\ & & & - & (326/45)d & = & -26080/9 \end{array}$$

The third equation tells us that $d = (-45/326)(-26080/9) = 400$. Substituting this into the second equation, we have $s+(8/9)(400) = 8600/9 \Rightarrow s = (8600/9)-(3200/9) = 600$. And substituting both of these into the first equation yields $a + (1/2)(600) + (2/5)(400) = 760 \Rightarrow A = 760 - 300 - 160 = 300$. So, we have found that armchairs cost \$300, sofa beds cost \$600, and double beds cost \$400.

B9 Let b be the average mark for the business students, l be the average mark for the liberal arts students, and s be the average mark for the science students. Then to determine the average for each group of students, we need to solve the following system of equations:

$$\begin{aligned} \tfrac{b+l+s}{3} &= 85 \\ \tfrac{100b+300l+200s}{600} &= 86 \\ \tfrac{100b+200s}{300} &= l+4 \end{aligned}$$

Let's put these equations into our standard linear equation form:

$$\begin{array}{rcrcrcr} (1/3)b & + & (1/3)l & + & (1/3)s & = & 85 \\ (1/6)b & + & (1/2)l & + & (1/3)s & = & 86 \\ (1/3)b & - & l & + & (2/3)s & = & 4 \end{array}$$

And now we can create an augmented matrix for our system:

$$\left[\begin{array}{ccc|c} 1/3 & 1/3 & 1/3 & 85 \\ 1/6 & 1/2 & 1/3 & 86 \\ 1/3 & -1 & 2/3 & 4 \end{array}\right]$$

The next step is to find a row equivalent matrix in row echelon form:

$$\left[\begin{array}{ccc|c} 1/3 & 1/3 & 1/3 & 85 \\ 1/6 & 1/2 & 1/3 & 86 \\ 1/3 & -1 & 2/3 & 4 \end{array}\right] \begin{array}{c} 3R_1 \\ 6R_2 \\ 3R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 255 \\ 1 & 3 & 2 & 516 \\ 1 & -3 & 2 & 12 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 255 \\ 0 & 2 & 1 & 261 \\ 0 & -4 & 1 & -243 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 2R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 255 \\ 0 & 2 & 1 & 261 \\ 0 & 0 & 3 & 279 \end{array}\right].$$

Putting the row echelon form back into equations, we get

$$\begin{array}{rcrcrcr} b & + & l & + & s & = & 255 \\ & & 2l & + & s & = & 261 \\ & & & & 3s & = & 279 \end{array}$$

The third equation tells us that $s = 93$. Substituting this into the second equation gives us $2l+93 = 261 \Rightarrow l = 84$. And substituting both of these values into the first equation yields $b+84+93 = 255 \Rightarrow b = 78$. So, we have found that the average for the students in business is 78%, the average for the students in liberal arts is 84%, and the average for the students in science is 93%.

a (i) $\left[\begin{array}{ccc|c} 2 & 6 & 7 & 6 \\ 3 & 9 & 3 & -6 \\ -2 & -4 & -6 & 5 \end{array}\right]$

(ii) $\left[\begin{array}{ccc|c} 2 & 6 & 7 & 6 \\ 3 & 9 & 3 & -6 \\ -2 & -4 & -6 & 5 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 3 & 9 & 3 & -6 \\ 2 & 6 & 7 & 6 \\ -2 & -4 & -6 & 5 \end{array}\right] \begin{array}{c} (1/3)R_1 \\ \\ \\ \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & -2 \\ 2 & 6 & 7 & 6 \\ -2 & -4 & -6 & 5 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array} \left(\begin{array}{ccc|c} -2(1 & 3 & 1 & -2) \\ \hline -2 & -6 & -2 & 4 \\ 2 & 6 & 7 & 6 \\ \hline 0 & 0 & 5 & 10 \end{array}\right)$

$\left(\begin{array}{ccc|c} 2(1 & 3 & 1 & -2) \\ \hline 2 & 6 & 2 & -4 \\ -2 & -4 & -6 & 5 \\ \hline 0 & 2 & -4 & 1 \end{array}\right) \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 2 & -4 & 1 \end{array}\right] \; R_2 \updownarrow R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & -2 \\ 0 & 2 & -4 & 1 \\ 0 & 0 & 5 & 10 \end{array}\right]$

(iii) The row echelon form matrix does not have any bad rows, so the system is consistent. As the REF matrix has three pivots and the system has three variables, the system has a unique solution, and therefore there are zero parameters in the general solution.

(iv) We begin by switching the REF matrix into equations:

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & + & x_3 & = & -2 \\ & & 2x_2 & - & 4x_3 & = & 1 \\ & & & & 5x_3 & = & 10 \end{array}$$

The third equation tells us that $x_3 = 2$. Substituting this into the second equation yields $2x_2 - 8 = 1 \Rightarrow x_2 = 9/2$. Substituting both of these into the first equation yields $x_1 + 27/2 + 2 = -2 \Rightarrow x_1 = -35/2$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -35/2 \\ 9/2 \\ 2 \end{bmatrix}$$

(b) (i) $\left[\begin{array}{cccc|c} 1 & -1 & 3 & 7 & -3 \\ 3 & -5 & 11 & 5 & 7 \\ -2 & 3 & -7 & -6 & 0 \end{array}\right]$

(ii) $\left[\begin{array}{cccc|c} 1 & -1 & 3 & 7 & -3 \\ 3 & -5 & 11 & 5 & 7 \\ -2 & 3 & -7 & -6 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array} \left(\begin{array}{cccc|c} -3(1 & -1 & 3 & 7 & -3) \\ \hline -3 & 3 & -9 & -21 & 9 \\ 3 & -5 & 11 & 5 & 7 \\ \hline 0 & -2 & 2 & -16 & 16 \end{array}\right)$

$$\left(\begin{array}{cccc|c} 2(1 & -1 & 3 & 7 & -3) \\ \hline 2 & -2 & 6 & 14 & -6 \\ -2 & 3 & -7 & -6 & 0 \\ \hline 0 & 1 & -1 & 8 & -6 \end{array}\right) \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 7 & -3 \\ 0 & -2 & 2 & -16 & 16 \\ 0 & 1 & -1 & 8 & -6 \end{array}\right] \quad R_3 + (1/2)R_2$$

$$\left(\begin{array}{cccc|c} (1/2)(0 & -2 & 2 & -16 & 16) \\ \hline 0 & -1 & 1 & -8 & 8 \\ 0 & 1 & -1 & 8 & -6 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right) \sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 7 & -3 \\ 0 & -2 & 2 & -16 & 16 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

(iii) Because the last row of the row echelon matrix is a bad row, the system is inconsistent.

(c) (i) $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 5 \\ 5 & -5 & -7 & 17 & -10 \\ -8 & 8 & 0 & -16 & -40 \\ -3 & 3 & 2 & -8 & -5 \end{array}\right]$

(ii) $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 5 \\ 5 & -5 & -7 & 17 & -10 \\ -8 & 8 & 0 & -16 & -40 \\ -3 & 3 & 2 & -8 & -5 \end{array}\right] \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 + 8R_1 \\ R_4 + 3R_1 \end{array} \left(\begin{array}{cccc|c} -5(1 & -1 & 0 & 2 & 5) \\ \hline -5 & 5 & 0 & -10 & -25 \\ 5 & -5 & -7 & 17 & -10 \\ \hline 0 & 0 & -7 & 7 & -35 \end{array}\right)$

$$\left(\begin{array}{cccc|c} 8(1 & -1 & 0 & 2 & 5) \\ \hline 8 & -8 & 0 & 16 & 40 \\ -8 & 8 & 0 & -16 & -40 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{cccc|c} 3(1 & -1 & 0 & 2 & 5) \\ \hline 3 & -3 & 0 & 6 & 15 \\ -3 & 3 & 2 & -8 & -5 \\ \hline 0 & 0 & 2 & -2 & 10 \end{array}\right)$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 5 \\ 0 & 0 & -7 & 7 & -35 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 10 \end{array}\right] \begin{array}{c} (-1/7)R_2 \\ R_3 \updownarrow R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 2 & -2 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 - 2R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(iii) The row echelon form matrix does not have any bad rows, so the system is consistent. As the REF matrix has two pivots, but the system has four variables, we know that the general solution will have two parameters.

(iv) We begin by switching the REF matrix into equations:

$$\begin{array}{ccccccccc} x_1 & - & x_2 & & & + & 2x_4 & = & 5 \\ & & & & x_3 & - & x_4 & = & 5 \end{array}$$

Next, we replace the variable x_2 with the parameter s and the variable x_4 with the parameter t. This turns our system into

$$\begin{array}{ccccccccc} x_1 & - & s & & & + & 2t & = & 5 \\ & & & & x_3 & - & t & = & 5 \end{array}$$

The second equation tells us that $x_3 = 5 + t$, while the second equation tells us that $x_1 = 5 + s - 2t$. So, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 + s - 2t \\ s \\ 5 + t \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$