Solution to Practice 2f

B4(a) The augmented matrix is already in row echelon form. As we see that it has no bad rows, the system is CONSISTENT. To find the general solution, we first translate the matrix back into a system:

$$\begin{array}{ccccccccc} x_1 & - & x_2 & + & x_3 & & & = & 1 \\ & & x_2 & & & - & x_4 & = & 2 \\ & & & & 2x_3 & + & 4x_4 & + & -3 \end{array}$$

and then do back-substitution. The first step in the back-substitution is to replace the variable x_4 with the parameter s, giving us

$$\begin{array}{ccccccccc} x_1 & - & x_2 & + & x_3 & & & = & 1 \\ & & x_2 & & & - & s & = & 2 \\ & & & & 2x_3 & + & 4s & + & -3 \end{array}$$

Then the third equation becomes $2x_3 = -3 - 4s$, so $x_3 = -(3/2) - 2s$. The second equation becomes $x_2 = 2+s$. And substituting these values for x_3 and x_2 into the first equation gives us $x_1 - (2+s) + (-(3/2) - 2s) = 1 \Rightarrow x_1 = 9/2 + 3s$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9/2 + 3s \\ 2 + s \\ -(3/2) - 2s \\ s \end{bmatrix} = \begin{bmatrix} 9/2 \\ 2 \\ -3/2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

B4(b) The third row is a bad row, so the system in INCONSISTENT.

B4(c) The augmented matrix is already in row echelon form. As we see that it has no bad rows, the system is CONSISTENT. To find the general solution, we first translate the matrix back into a system:

$$\begin{array}{ccccccccc} 2x_1 & + & x_2 & & & + & x_4 & = & 2 \\ & & x_2 & & & + & x_4 & = & 1 \\ & & & & x_3 & + & x_4 & = & 2 \end{array}$$

and then do back-substitution. The first step in the back-substitution is to replace the variable x_4 with the parameter s, giving us

$$\begin{array}{ccccccccc} 2x_1 & + & x_2 & & & + & s & = & 2 \\ & & x_2 & & & + & s & = & 1 \\ & & & & x_3 & + & s & + & 2 \end{array}$$

Then the third equation tells us $x_3 = 2 - s$ and the second equation tells us $x_2 = 1 - s$. Substituting these values into the first equation yields $2x_1 + (1 - s) + s = 2 \Rightarrow 2x_1 + 1 = 2 \Rightarrow x_1 = 1/2$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1-s \\ 2-s \\ s \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

B4(d) The augmented matrix is already in row echelon form. As we see that it has no bad rows, the system is CONSISTENT. (Remember: rows of all zero are fine, and so are non-zero entries = 0. Only 0 entries = non-zero are bad.) To find the general solution, we first translate the matrix back into a system:

$$\begin{array}{ccccccc} x_1 & + & x_3 & & & = & -1 \\ & & x_3 & - & x_4 & = & 0 \end{array}$$

and then do back-substitution. (At this point, there is no value to the $0 = 0$ equations, so I've left them out.) The first step in the back-substitution is to replace the variables x_2 and x_4 with the parameter s and t respectively, giving us

$$\begin{array}{ccccccc} x_1 & + & x_3 & & & = & -1 \\ & & x_3 & - & t & = & 0 \end{array}$$

(Yes, I realize that there are no x_2 terms at this point, but we will still have one in our solution, and that's where we'll plug in $x_2 = s$.) The second equation tells us that $x_3 = t$, and substituting this into the first equation yields $x_1 + t = -1 \Rightarrow x_1 = -1 - t$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

B5(a)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

Since this matrix has no bad rows, the system is CONSISTENT. Moreover, since there are three variables but only two pivots, we know that the general solution will have one parameter.

Moving on to part (iv), we turn this matrix into the system

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & -2 \\ & - & x_2 & + & 3x_3 & = & 0 \end{array}$$

Next, we need to replace the variable x_3 with the parameter s, giving us

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & s & + & -2 \\ & - & x_2 & + & 3s & = & 0 \end{array}$$

Then the second equation tells us that $-x_2 = -3s$, so $x_2 = 3s$. Substituting this into the first equation yields $x_1 + 3s + s = -2 \Rightarrow x_1 = -2 - 4s$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-4s \\ 3s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

B5(b)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since this matrix has no bad rows, the system in CONSISTENT. Moreover, since there are three variables but only two pivots, we know that the general solution will have one parameter.

Moving on to part (iv), we turn this matrix into the system

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & - & 2x_3 & = & 1 \\ & & 5x_2 & + & 3x_3 & = & 4 \end{array}$$

Next, we need to replace the variable x_3 with the parameter s, giving us

$$\begin{array}{ccccccc} x_1 & - & 2x_2 & - & 2s & = & 1 \\ & & 5x_2 & + & 3s & = & 4 \end{array}$$

Then the second equation becomes $5x_2 = 4 - 3s$, so $x_2 = (4/5) - (3/5)s$. Substituting this value into the first equation yields $x_1 - 2((4/5) - (3/5)s) - 2s = 1 \Rightarrow x_1 - 8/5 - (4/5)s = 1 \Rightarrow x_1 = (13/5) + (4/5)s$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (13/5) + (4/5)s \\ (4/5) - (3/5)s \\ s \end{bmatrix} = \begin{bmatrix} 13/5 \\ 4/5 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4/5 \\ -3/5 \\ 1 \end{bmatrix}$$

B5(c)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Because the last row is a bad row, the system is INCONSISTENT. There is no step (iv).

B5(d)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & -1 & -34 \end{array}\right]$$

Since this matrix has no bad rows, the system in CONSISTENT. Since the system has both three variables and three pivots, it will have a unique solution, and therefore will have zero parameters.

Moving on to part (iv), we turn this matrix into the system

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & = & -7 \\ & & x_2 & + & x_3 & = & 16 \\ & & & - & x_3 & = & -34 \end{array}$$

The third equation tells us that $x_3 = 34$. Plugging this into the second equation tells us that $x_2 + 34 = 16 \Rightarrow x_2 = -18$. And finally, plugging this value into the first equation tells us that $x_1 - 18 = -7 \Rightarrow x_1 = 11$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -18 \\ 34 \end{bmatrix}$$

B5(e)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & -20 \end{array}\right]$$

Because the last row is a bad row, the system is INCONSISTENT. There is no step (iv).

B5(f)(iii),(iv) In the last lecture, we found the following row echelon form for the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since this matrix has no bad rows, the system in CONSISTENT. Since the system has four varialbes, but only two pivots, the general solution will have two parameters.

Moving on to part (iv), we turn this matrix into the system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 1 \\ & & x_2 & + & 2x_3 & & & = & -2 \end{array}$$

Next, we need to replace the variables x_3 and x_4 with the parameters s and t, respectively, giving us

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & 2s & + & t & = & 1 \\ & & x_2 & + & 2s & & & = & -2 \end{array}$$

Then the second equation tells us that $x_2 = -2 - 2s$. Substituting this into the first equation yields $x_1 + (-2 - 2s) + 2s + t = 1 \Rightarrow x_1 - 2 + t = 1 \Rightarrow x_1 = 3 - t$. So the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3-t \\ -2-2s \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

B6(a) The first thing we notice about the matrix is that we need $b \neq 0$, so that the bottom row is not a bad row. Next, we turn our attention to the $a^2 - 1$ entry. If $a^2 - 1 \neq 0$, then we can divide row 2 by $a^2 - 1$ and row 3 by b to get a matrix that looks like:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

where the "*" are not necessarily the same. It is easy enough to calculate the values of "*", but I omit them because it doesn't matter what they are! From the information we have we know that we have a system with a unique solution.

But what happens if $a^2 - 1 = 0$. That is, what happens if $a = 1$ or $a = -1$. We need to look at these cases separately.

If $a = 1$, then our matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & b & -3 \end{array}\right]$$

To get the matrix into row echelon form, we subtract b times row 2 from row 3 to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3-3b \end{array}\right]$$

To have this system be consistent, we need to have $-3 - 3b = 0 \Rightarrow b = -1$. In this scenario, our system will not have a unique solution.

Similarly, if $a = -1$, then our matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & b & -3 \end{array}\right]$$

To get the matrix into row echelon form, we add b times row 2 from row 3 to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -3+3b \end{array}\right]$$

To have this system be consistent, we need to have $-3 + 3b = 0 \Rightarrow b = 1$. In this scenario, our system will not have a unique solution.

So, to summarize our results: In order to have a unique solution, we need $a \neq 1, -1, b \neq 0$. Our system is also consistent if $a = 1, b = -1$ and if $a = -1, b = 1$. Otherwise, the system is inconsistent.

B6(b) Looking at row 2, we see that we need $c \neq 0$. Next, we turn our attention to the last row. If we let $d = 0$, the last row becomes "$0\ 0\ 0\ 0 \mid -c$", and since $c \neq 0$, this would be a bad row. Thus, we also need $d \neq 0$. This also means that $cd \neq 0$, so we can divide rows two and three by c and row 4 by cd to get the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 5 & 2 \\ 0 & 1 & 1 & 0 & 1/c \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & (c+d)/cd \end{array}\right]$$

From this we see that the system is consistent, with a unique solution.

So, if $c, d \neq 0$, the system is consistent and has a unique solution. Otherwise the system is inconsistent.

D1 First, lets put this system in matrix form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 2 & 3 & 1 & 5 & 6 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & a & 1 \end{array}\right]$$

Next, we want to put this matrix into row echelon form. I see that I already have a 1 in the first column of row 1, and that all that remains to establish this pivot is to add -2 times row 1 to row 2. The calculation for this is

$$\begin{array}{cccc|c} -2(1 & 1 & 0 & 1 & b) \\ \hline -2 & -2 & 0 & -2 & -2b \\ 2 & 3 & 1 & 5 & 6 \\ \hline 0 & 1 & 1 & 3 & 6-2b \end{array}$$

and we now look at the row equivalent matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & a & 1 \end{array}\right]$$

Now we have a 1 in the second column of the second row, so all that is left to establish the pivot in the second row is to add -2 times row 2 to row 4. The calculation for this is

$$\begin{array}{cccc|c} -2(0 & 1 & 1 & 3 & 6-2b) \\ \hline 0 & -2 & -2 & -6 & -12+4b \\ 0 & 2 & 2 & a & 1 \\ \hline 0 & 0 & 0 & a-6 & -11+4b \end{array}$$

and we now look at the row equivalent matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & a-6 & -11+4b \end{array}\right]$$

This matrix is in row echelon form, so we can now ponder the questions of consistency and uniqueness.

(a) This system will be inconsistent if and only if the previous matrix has a bad row. Rows 1, 2, and 3 cannot possibly be bad rows, but row 4 could be. First, we would need all the entries on the left to be zeros, which means we need $a - 6 = 0 \Rightarrow a = 6$. Second, we need the entry on the right to be non-zero, which means that $-11 + 4b \neq 0 \Rightarrow b \neq 11/4$. So, this system is inconsistent if $a = 6$ and $b \neq 11/4$.

(b) For the system to be consistent with a unique solution, we need to have a pivot in every row of the coefficient matrix. We definitely have a pivot in rows 1, 2, and 3, so we simply need to make sure that we have a pivot in row 4 of the coefficient matrix. That is, we need $a - 6 \neq 0$. So, the system is consistent with a unique solution when $a \neq 6$. (Notice that in this case, we don't care what -11 + $4b$ is.)

(c) Since we definitely have a pivot in rows 1, 2, and 3 of the coefficient matrix, the only way for there to be infinitely many solutions is for there to NOT be a pivot in row 4. But to ensure that there are no bad rows, we need row 4 of the augmented matrix to be a row of all zeros. This means we need $a - 6 = 0$ (that is, $a = 6$), and we need $-11 + 4b = 0$, (that is, $b = 11/4$). So, when $a = 6$ and $b = 11/4$, this system is consistent with infinitely many solutions.