Solution to Practice 2e

Note on row echelon form: The row echelon form of a matrix is not unique, and therefore you may get a different answer than the one I give in my solution.

B2 A, B, and D are in row echelon form. C is not in row echelon form, because the first non-zero entry in the first row is not to the left of the first non-zero entry in the second row.

B3(a)

$$\begin{bmatrix} 2 & 4 & 2 & -1 \\ 1 & 3 & 3 & 0 \\ 2 & 5 & 5 & -4 \end{bmatrix} \begin{array}{l} R_1 \updownarrow R_2 \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 3 & 3 & 0 \\ 2 & 4 & 2 & -1 \\ 2 & 5 & 5 & -4 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left(\begin{array}{cccc} -2(1 & 3 & 3 & 0) \\ \hline -2 & -6 & -6 & 0 \\ 2 & 4 & 2 & -1 \\ \hline 0 & -2 & -4 & -1 \end{array}\right) \left(\begin{array}{cccc} -2(1 & 3 & 3 & 0) \\ \hline -2 & -6 & -6 & 0 \\ 2 & 5 & 5 & -4 \\ \hline 0 & -1 & -1 & -4 \end{array}\right) \sim \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & -2 & -4 & -1 \\ 0 & -1 & -1 & -4 \end{bmatrix} \ R_2 \updownarrow R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & -1 & -1 & -4 \\ 0 & -2 & -4 & -1 \end{bmatrix} \ R_3 - 2R_2 \quad \left(\begin{array}{cccc} -2(0 & -1 & -1 & -4) \\ \hline 0 & 2 & 2 & 8 \\ 0 & -2 & -4 & -1 \\ \hline 0 & 0 & -2 & 7 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & -2 & 7 \end{bmatrix}$$

B3(b)

$$\begin{bmatrix} 5 & 7 & -2 & 7 & 4 \\ 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & -6 & 5 \end{bmatrix} \begin{array}{l} R_1 \updownarrow R_2 \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 5 & 7 & -2 & 7 & 4 \\ 3 & 0 & 3 & -6 & 5 \end{bmatrix} \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccccc} -5(1 & 1 & 0 & 1 & 0) \\ \hline -5 & -5 & 0 & -5 & 0 \\ 5 & 7 & -2 & 7 & 4 \\ \hline 0 & 2 & -2 & 2 & 4 \end{array}\right) \left(\begin{array}{ccccc} -3(1 & 1 & 0 & 1 & 0) \\ \hline -3 & -3 & 0 & -3 & 0 \\ 3 & 0 & 3 & -6 & 5 \\ \hline 0 & -3 & 3 & -9 & 5 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 2 & 4 \\ 0 & -3 & 3 & -9 & 5 \end{bmatrix} \ R_3 + (3/2)R_2 \quad \left(\begin{array}{ccccc} 3/2(0 & 2 & -2 & 2 & 4) \\ \hline 0 & 3 & -3 & 3 & 6 \\ 0 & -3 & 3 & -9 & 5 \\ \hline 0 & 0 & 0 & -6 & 11 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 2 & 4 \\ 0 & 0 & 0 & -6 & 11 \end{bmatrix}$$

B3(c)

$$\begin{bmatrix} 2 & 1 & 0 & 7 \\ 1 & 1 & -1 & 2 \\ 3 & 2 & 0 & 12 \\ 6 & 4 & -1 & 25 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 0 & 7 \\ 3 & 2 & 0 & 12 \\ 6 & 4 & -1 & 25 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 6R_1 \end{matrix}$$

$$\left(\begin{array}{cccc} -2(1 & 1 & -1 & 2) \\ \hline -2 & -2 & 2 & -4 \\ 2 & 1 & 0 & 7 \\ \hline 0 & -1 & 2 & 3 \end{array}\right) \left(\begin{array}{cccc} -3(1 & 1 & -1 & 2) \\ \hline -3 & -3 & 3 & -6 \\ 3 & 2 & 0 & 12 \\ \hline 0 & -1 & 3 & 6 \end{array}\right) \left(\begin{array}{cccc} -6(1 & 1 & -1 & 2) \\ \hline -6 & -6 & 6 & -12 \\ 6 & 4 & -1 & 25 \\ \hline 0 & -2 & 5 & 13 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -1 & 3 & 6 \\ 0 & -2 & 5 & 13 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{matrix} \left(\begin{array}{cccc} -1(0 & -1 & 2 & 3) \\ \hline 0 & 1 & -2 & -3 \\ 0 & -1 & 3 & 6 \\ \hline 0 & 0 & 1 & 3 \end{array}\right) \left(\begin{array}{cccc} -2(0 & -1 & 2 & 3) \\ \hline 0 & 2 & -4 & -6 \\ 0 & -2 & 5 & 13 \\ \hline 0 & 0 & 1 & 7 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

B3(d)

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 2 & 5 & 2 & 6 & 1 \\ 3 & 7 & 4 & 9 & 3 \\ 2 & 6 & 2 & 6 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{matrix} \left(\begin{array}{ccccc} -2(1 & 2 & 1 & 3 & 0) \\ \hline -2 & -4 & -2 & -6 & 0 \\ 2 & 5 & 2 & 6 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccccc} -3(1 & 2 & 1 & 3 & 0) \\ \hline -3 & -6 & -3 & -9 & 0 \\ 3 & 7 & 4 & 9 & 3 \\ \hline 0 & 1 & 1 & 0 & 3 \end{array}\right) \left(\begin{array}{ccccc} -2(1 & 2 & 1 & 3 & 0) \\ \hline -2 & -4 & -2 & -6 & 0 \\ 2 & 6 & 2 & 6 & 5 \\ \hline 0 & 2 & 0 & 0 & 5 \end{array}\right)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

B5(ii)(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 5 & -4 \\ 1 & 1 & 1 & -2 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 2 & 1 & 5 & -4 \end{array}\right] \; R_2 - 2R_1 \; \left(\begin{array}{ccc|c} -2(1 & 1 & 1 & -2) \\ \hline -2 & -2 & -2 & 4 \\ 2 & 1 & 5 & -4 \\ \hline 0 & -1 & 3 & 0 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

B5(ii)(b)

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & -2 & -2 & 1 \\ -1 & 12 & 8 & 7 \end{array}\right] \; R_1 \updownarrow R_2 \; \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 2 & 1 & -1 & 6 \\ -1 & 12 & 8 & 7 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} -2(1 & -2 & -2 & 1) \\ \hline -2 & 4 & 4 & -2 \\ 2 & 1 & -1 & 6 \\ \hline 0 & 5 & 3 & 4 \end{array}\right) \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 10 & 6 & 8 \end{array}\right] \; R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

B5(ii)(c)

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 9 \end{array}\right] \; R_1 \updownarrow R_2 \; \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 9 \end{array}\right] \; R_3 - 2R_1 \; \left(\begin{array}{ccc|c} -2(1 & 1 & 1 & 3) \\ \hline -2 & -2 & -2 & -6 \\ 2 & 3 & 3 & 9 \\ \hline 0 & 1 & 1 & 3 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \; R_3 - R_2 \; \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

B5(ii)(d)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 2 & 4 & 1 & -16 \\ 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left(\begin{array}{ccc|c} -2(1 & 1 & 0 & -7) \\ \hline -2 & -2 & 0 & 14 \\ 2 & 4 & 1 & -16 \\ \hline 0 & 2 & 1 & -2 \end{array}\right) \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 1 & 16 \end{array}\right] \; R_2 \updownarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 1 & 16 \\ 0 & 2 & 1 & -2 \end{array}\right] \; R_3 - 2R_2 \; \left(\begin{array}{ccc|c} -2(0 & 1 & 1 & 16) \\ \hline 0 & -2 & -2 & -32 \\ 0 & 2 & 1 & -2 \\ \hline 0 & 0 & -1 & -34 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & -1 & -34 \end{array}\right]$$

B5(ii)(e)

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 1 & 2 & 4 & 1 & 7 \\ 1 & 0 & 0 & 1 & -21 \end{array}\right] \begin{array}{c} R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & -1 & -2 & 0 & -24 \end{array}\right] \; R_3 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & -20 \end{array}\right]$$

B5(ii)(f)

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & -1 & -2 & 0 & 2 \end{array}\right] \ R_3 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Comment on (e) and (f): Notice that I performed the exact same row operations for (e) and (f). This is because they have the same coefficient matrix. In fact, if you look at the solution to (e) and the solution to (f), the only place you will find a difference is in the last column of the matrices. In general, we could have row reduced these at the same time, using something of a "double augmented matrix" like

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 2 & 1 & 3 & 1 \\ 1 & 2 & 4 & 1 & 7 & -1 \\ 1 & 0 & 0 & 1 & -21 & 3 \end{array}\right]$$

And if we had even more systems with the same coefficient matrix, we could just keep on augmenting the right side onto this matrix, and solve all these related systems at the same time. We will make use of this ability later in the course.