Lecture 2g

Solving a System of Linear Equations

(Summarizing the results of Section 2.1)

So far, the process of solving a system of linear equations has been broken into little pieces, so I thought that we should end our discussion of Section 2.1 by putting all the pieces together! To that end, let's solve the following word problem.

Example: A woman has a coin purse that contains toonies, loonies, quarters, dimes, and nickels. The toonies and loonies have a combined value of \$4, while the remaining coins have a combined value of \$2.40. There are a total of 17 coins, and there is one more quarter than there are dimes. How many of each type of coin are in the coin purse?

First, we need to set this up as a system of linear equations. Let x_1 be the number of nickels, x_2 be the number of dimes, x_3 be the number of quarters, x_4 be the number of loonies, and x_5 be the number of toonies.

The fact that she has a total of 17 coins becomes the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 17$$

The fact that there is one more quarter than there are dimes is expressed by the equation $x_3 = x_2 + 1$, but we will bring the x_2 over to the other side to put the equation into the standard form of a linear equations, getting

$$-x_2 + x_3 = 1$$

The fact that the toonies and loonies have a combined value of \$4 gives us the following linear equation

$$x_4 + 2x_5 = 4$$

And finally, the fact that the quarters, dimes, and nickels have a combined value of \$2.40 gives us the following linear equation

$$(.05)x_1 + (.1)x_2 + (.25)x_3 = 2.40$$

And so, to determine how many of each type of coin are in the coin purse, we need to solve the following system of equations:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 17$$
$$-x_2 + x_3 = 1$$
$$x_4 + 2x_5 = 4$$
$$(.05)x_1 + (.1)x_2 + (.25)x_3 = 2.40$$

The first step is to write the augmented matrix for this system:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ .05 & .1 & .25 & 0 & 0 & 2.40 \end{array}\right]$$

Next, we need to row reduce this matrix to row echelon form. To that end, I am first going to multiply the fourth row by 20 to clear out the decimals and multiply the second row by -1 to make its leading entry a 1. Neither of these are necessary, but I find them useful. This gives us

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 1 & 2 & 5 & 0 & 0 & 48 \end{array}\right]$$

And now the real work begins. We already have a "1" in the first column of the first row, so to establish this pivot we simply need to eliminated any non-zero entries beneath this row. It happens that there is only one such entry, in the fourth row, so if we replace row 4 with row 4 minus row 1 we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & -1 & -1 & 31 \end{array}\right]$$

Now that we have established a pivot in row 1, we need to establish a pivot in row 2. Luckily, the second column of row 2 already contains a non-zero entry, so all we need to do is eliminate all the non-zero entries in column two below row 2. As before, the only such entry is in row 4, so if we replace row 4 with row 4 minus row 2 we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & -1 & -1 & 32 \end{array}\right]$$

Now that we have established a pivot in row 2, we need to establish a pivot in row 3. The first column with a non-zero entry below row 2 is column 3, which has a non-zero entry in row 4. So, if we interchange row 3 and row 4, we get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 5 & -1 & -1 & 32 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

We now have a matrix in row echelon form, so this stage of solving the system is complete. From the row echelon form, we see that the system is consistent, and that the general solution will have one parameter. This parameter will correspond to the variable "x_5", as column 5 does not contain a pivot. So, our next step is to put our augmented matrix back into equation form, but replacing the variable x_5 with the parameter s. This gives us the following:

$$\begin{array}{l} x_1 + x_2 + x_3 + x_4 + s = 17 \\ x_2 - x_3 = -1 \\ 5x_3 - x_4 - s = 32 \\ x_4 - 2s = 4 \end{array}$$

We begin the back-substitution process by looking at the fourth equation:

$$x_4 - 2s = 4 \Rightarrow x_4 = 4 - 2s$$

We substitute this value for x_4 into our third equation, getting:

$$\begin{array}{rcl} 5x_3 - x_4 - s = 32 & \Rightarrow & 5x_3 - (4 - 2s) - s = 31 \\ & \Rightarrow & 5x_3 = 36 - s \\ & \Rightarrow & x_3 = (36/5) - (1/5)s \end{array}$$

We now substitute the above value of x_3 into our second equation, getting:

$$\begin{array}{rcl} x_2 - x_3 = -1 & \Rightarrow & x_2 - ((36/5) - (1/5)s) = -1 \\ & \Rightarrow & x_2 = (31/5) - (1/5)s \end{array}$$

Finally, we substitute the above values of x_4, x_3, and x_2 into our first equation, getting

$$\begin{array}{rcl} x_1 + x_2 + x_3 + x_4 + s = 17 & \Rightarrow & x_1 + (31/5) - (1/5)s + (36/5) - (1/5)s + 4 - 2s + s = 17 \\ & \Rightarrow & x_1 = (-2/5) + (7/5)s \end{array}$$

From this, we see that the general solution to the system of linear equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (-2/5) + (7/5)s \\ (31/5) - (1/5)s \\ (36/5) - (1/5)s \\ 4 - 2s \\ s \end{bmatrix} = \begin{bmatrix} -2/5 \\ 31/5 \\ 36/5 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 7/5 \\ -1/5 \\ -1/5 \\ -2 \\ 1 \end{bmatrix}$$

But this is where this example gets interesting. Because this "general solution" doesn't actually make sense when we consider the original question: How many coins does the woman have in her coin purse? She can't possibly have $-2/5$ nickels in her coin purse! So, while the system of linear equations has an infinite number of solutions, the word problem actually only has one solution–we just need to use some common sense to find it. First of all, let's recall that s represents the number of toonies we have, so it must be a non-negative integer. Moreover, if we recall that the total value of loonies and toonies is \$4, we know that there can be at most two toonies in the coin purse. So we have narrowed down the possibilities for s to 0, 1, or 2. We've already pointed out that s cannot be 0, since this results in $x_1 = -2/5$, which would mean that there were $-2/5$ nickels in the coin purse. Similarily, if s is 2 then we also end up needing the impossible number of 12/5 nickels in the coin purse. But, when s=1, we find that $x_1 = 1$, $x_2 = 6$, $x_3 = 7$, $x_4 = 2$, and $x_5 = 1$, which gives us the only possible answer: that there are 1 nickel, 6 dimes, 7 quarters, 2 loonies, and 1 toonie in the coin purse.

The reason that I like this word problem is that it gives me the opportunity to remind students that math is more than just about numbers and calculations. If we had simply stopped after finding the general solution to the system of linear equations, we wouldn't have solved our problem! So, one piece of advice I always give students is that, once you have finished your calculations, make sure you go back and read the problem again to make sure that you have actually answered the question!

This is also a prime example of why it is important for people to know math. Because, fundamentally, there are many computer programs that can solve a system of linear equations these days. But what you need to understand is WHEN to plug in to a computer, WHAT to plug in to a computer, and HOW TO INTERPRET the results the computer gives you.

Now, the realities of this course is that you will not have access to a computer for your final exam, so doing the calculations will be a part of your mark for this course. I will endeavor to make these calculations easy, even though this ends up obscuring the value in certain techniques. But I don't think you really want me to give you a 10×10 matrix to row reduce by hand just to prove to you that matrices really are better! So, when I ask you to use matrix form and put something into row echelon form, please follow these directions instead of solving the system some clever way, because the only alternative to giving you easy row reductions is to give you hard row reductions, and none of us want that!