Lecture 2f

Consistent? Unique?

(pages 75-76)

While finding the general solution to a system of linear equations is always nice, sometimes all we really care about is whether or not there *is* a solution.

Definition: A system of linear equations that has at least one solution is called **consistent**. A system that does not have any solutions is called **inconsistent**.

It happens that we can determine whether or not a system is consistent by looking at a row echelon form of its augmented matrix. That is to say, we can determine whether or not a system is consistent BEFORE we do any back-substitution. But then, we new that already, since we know that the only way a system will have no solutions is if the the augmented matrix is row equivalent to a matrix with a row of the form $0\ 0 \cdots 0|c$ for $c \neq 0$. I like to refer to such rows as **bad rows**. So a system is INCONSISTENT if the row echelon form of the augmented matrix contains a bad row, and a system is CONSISTENT if the row echelon form does not contain any bad rows. While we may be able to determine that a system in inconsistent before we reach row echelon form, we need to row reduce all the way to a row echelon form to guarantee consistency. After all, how else would we know if a bad row was going to appear?

The row echelon form can tell us more than simply whether or not a system is consistent or inconsistent. It can also tell us the number of parameters in the general solution when the system is consistent. If you recall from Lecture 2b, we turn the variables that are not in a leading term into parameters. Well, the leading terms are now the pivots in our row echelon form, so we see that every column in the COEFFICIENT matrix that does not contain a pivot corresponds to a variable that is not a leading variable, and thus corresponds to a parameter in the general solution.

These results are summarized by the following:

Theorem 2.1.1: Suppose that the augmented matrix $[A|\vec{b}]$ of a system of linear equations is row equivalent to $[S|\vec{c}]$, which is in row echelon form.

(1) The given system is inconsistent if and only if some row of $[S|\vec{c}]$ is of the form $[0\ 0\ \cdots\ 0\ |c]$ with $c \neq 0$.

(2) If the system is consistent, then there are two possibilities. Either the number of pivots in S is equal to the number of variables in the system and thus the system has a unique solution, or the number of pivots is less than the number of variables and the system has infinitely many solutions.

(3) If the system has infinitely many solutions, the number of parameters in the general solution equals "number of variables in system"-"number of pivots in S".

Course Author's note: I added line (3) to Theorem 2.1.1 myself–it does not appear in the version in the textbook.

Example: The matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & -1 & 3 \\ 0 & 0 & 1 & b \end{array}\right]$$

is the augmented matrix of a system of linear equations. For what values of the constants a and b is the system (a) inconsistent, (b) consistent with infinitely many solutions, or (c) consistent with a unique solution?

Well, at first glance, it doesn't seem like there is a way to create a bad row in this matrix. But if we set $a = 0$, then we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & b \end{array}\right]$$

To put this into row echelon form, we need to add row 2 to row 3, giving us the row equivalent matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & b+3 \end{array}\right]$$

From this, we see that if $b + 3 \neq 0$, then the last row is a bad row, and thus the system is inconsistent. But if $b + 3 = 0$, then our matrix would be

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and this would correspond to a consistent system with infinitely many solutions. Finally, if we go back to the situation where $a \neq 0$, then our original matrix is already in row echelon form, and has three pivots. Since this is the same as the number of variables, this corresponds to the situation where the system is consistent with a unique solution. Summarizing these findings, we see that

(a) The system is inconsistent when $a = 0$ and $b \neq -3$.

(b) The system is consistent with infinitely many solutions when $a = 0$ and $b = -3$.

(c) The system is consistent with a unique solution when $a \neq 0$. (b can be anything)