Lecture 2e

Row Echelon Form

(pages 73-74)

At the end of Lecture 2a I said that we would develop an algorithm for solving a system of linear equations, and now that we have our matrix notation, we can proceed with that plan. The first thing we are going to do is pay attention to our goal. In Lecture 2b we looked at the back-substitution procedure, which gave us some idea of where we would like to end up. But "some idea" isn't good enough for an algorithm, so we now define the following:

Definition: A matrix is in **row echelon form (REF)** if

(1) When all entries in a row are zeros, this row appears below all rows that contain a non-zero entry.

(2) When two non-zero rows are compared, the first non-zero entry (called the leading entry), in the upper row is to the left of the leading entry in the lower row.

Note that it follows from these properties that all entries in a column beneath a leading entry must be 0.

Definition: The leading entry in a non-zero row of a matrix in reduced echelon form in known as a **pivot**.

Example: The matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right]$$

is NOT in row echelon form, because when we compare the second and third rows, the leading entry in the second row is not to the left of the leading entry in the third row. We also note that there is a non-zero entry below the leading entry of the second row. But, if we interchange the second and third rows, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right] \; R_2 \updownarrow R_3 \; \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -12 & -48 \end{array}\right]$$

and the new matrix $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -12 & -48 \end{array}\right]$ IS in row echelon form.

Example: The matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 4 & 13 \\ 0 & 0 & 1 & -3 & 8 \end{array}\right]$$

is in row echelon form.

These examples are taken from Lecture 2b, to emphasize the connection between row echelon form, and the situation we had in Lecture 2b where the elimination steps had been completed and all that was left were the back-substitution steps. Using our new terminology for matrices, our first goal will be to row reduce our augmented matrix until it is in row echelon form, and then we will translate our matrix back into a system of equations and use back-substitution to solve the system.

With our goal defined, we now begin the procedure for getting there. The textbook gives a description of how to get to row echelon form, and I shall provide my own description here, in the hopes that at least one of them will makes sense to you.

<u>Step 0–Is the matrix already in row echelon form</u>? If yes, you're done! If not, continue to the next step.

<u>Step 1–Establish a pivot in the first row</u>. To do this, we will identify the first COLUMN that contains a non-zero entry. (Usually this is the first column, but not always.) Let's call it Column A. Interchanging rows if necessary, make the entry in Column A of the first row non-zero. This will be the pivot for row 1. Then use elementary row operations of type (3) to make all of the entries in Column A below the first row equal to zero. (Because of the calculations involved in this step, you will find it convenient to have the pivot be 1. So when you are establishing a pivot in row 1 you might want to look for a row that has a 1 in Column A, or failing that you can look for a row whose entries are all a multiple of their entry in Column A, so that you can divide by that entry and get a 1 in Column A. This won't always be possible, however.) If the matrix is now in row echelon form, you're done. Otherwise, continue to the next step.

Before we discuss Step 2, let's look at an example of Step 1:

Example: Consider the matrix

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Step 0: This matrix is not in row echelon form, since the leading entry in the first row is not to the left of the leading entry in the second row. So we move on to...

Step 1: The first column contains non-zero entries, so we will interchange rows to make the entry in the first column of the first row non-zero. Seeing that the first entry in row 3 is a 1, I will interchange row 1 and row 3, getting

$$\left[\begin{array}{ccccc|c} 0 & 4 & 8 & 0 & 3 & 0 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 1 & 2 & 5 & -1 & 0 & -1 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_3 \\ \\ \sim \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

Now I need to eliminate the "3" from the first column of the second row...

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \quad R_2 - 3R_1 \quad \left(\begin{array}{ccccc|c} -3(1 & 2 & 5 & -1 & 0 & -1) \\ \hline -3 & -6 & -15 & 3 & 0 & 3 \\ 3 & 7 & 17 & -3 & 2 & 2 \\ \hline 0 & 1 & 2 & 0 & 2 & 5 \end{array}\right)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

and the "5" from the first column of the fourth row...

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \quad R_4 - 5R_1 \quad \left(\begin{array}{ccccc|c} -5(1 & 2 & 5 & -1 & 0 & -1) \\ \hline -5 & -10 & -25 & 5 & 0 & 5 \\ 5 & 11 & 27 & -5 & 4 & 8 \\ \hline 0 & 1 & 2 & 0 & 4 & 13 \end{array}\right)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right]$$

and the "-1" from first column of the fifth row...

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ -1 & 1 & 1 & 1 & 6 & 4 \end{array}\right] \quad R_5 + R_1 \quad \left(\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 & 6 & 4 \\ \hline 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right]$$

We have now established a pivot in row 1, but the resulting matrix is not in row echelon form, as the leading entry in the second row is not to the left of the leading entry in the third row. So we will move on to the next step.

Step 2–Establish a pivot in the second row. From this point on we will ignore row 1. So, let Column B be the first column that has a non-zero entry in any row strictly below row 1. (Note that Column B will be to the right of Column A, since in Step 1 we made certain that every entry in Column A that is below row 1 was equal to 0.) Interchange rows if necessary to make the entry in column B of row 2 non-zero. As before, it is desirable to have this entry be a 1, since we will now proceed to use elementary row operations of type (3) to make every entry in Column B below row 2 equal to zero. If the matrix is now in row echelon form, you are done. Otherwise, move on to the next step.

Example continued:

Step 2: If we ignore row 1, we see that the first column with a non-zero entry is column 2. Row 2 already has a non-zero entry in column 2, and it is even a 1, so we do not need to interchange rows. Instead, we proceed to remove the "4" from row 3, the "1" from row 4, and the "3" from row 5. (I did these steps separately in Step 1, but this time I will do them all at the same time.)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ 0 & 3 & 6 & 0 & 6 & 3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 4R_2 \\ R_4 - R_2 \\ R_5 - 3R_2 \end{array} \left(\begin{array}{ccccc|c} -4(0 & 1 & 2 & 0 & 2 & 5) \\ \hline 0 & -4 & -8 & 0 & -8 & -20 \\ 0 & 4 & 8 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & -5 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} -1(0 & 1 & 2 & 0 & 2 & 5) \\ \hline 0 & -1 & -2 & 0 & -2 & -5 \\ 0 & 1 & 2 & 0 & 4 & 13 \\ \hline 0 & 0 & 0 & 0 & 2 & 8 \end{array}\right) \left(\begin{array}{ccccc|c} -3(0 & 1 & 2 & 0 & 2 & 5) \\ \hline 0 & -3 & -6 & 0 & -6 & -15 \\ 0 & 3 & 6 & 0 & 6 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

At this point, we have eliminated many of our entries, but the resulting matrix is still not in row echelon form, as the leading entry in row 3 is not to the left of the leading entry in row 4. So we will move on to the next step.

Step i–Establish a pivot in row i. Here is a generic description of a step. By the time you reach step i, you will have established a pivot in each of rows 1 to $i-1$. We may then ignore these rows. Let Column I be the first column that has a non-zero entry in a row below row $i-1$. Interchange rows if necessary to make the entry in Column I of row i non-zero. Then use elementary row operations of type (3) to make every entry in Column I below row i equal to zero. If the matrix is now in row echelon form, you are done. Otherwise, proceed to step $i+1$.

Example continued:

Step 3: We now ignore row 1 and row 2, and we see that the first column with a non-zero entry in the remaining rows is column 5. Row 3 already has a non-zero entry in column 5, and we see that if we multiply row 3 by -1/5, we can get the leading entry to be a 1:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & -5 & -20 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \quad (-1/5)R_3 \sim \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

To complete step 3, we simply need to eliminate the "2" from row 4:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \quad R_4 - 2R_3 \quad \sim \left(\begin{array}{ccccc|c} -2(0 & 0 & 0 & 0 & 1 & 4) \\ \hline 0 & 0 & 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 & 2 & 8 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right]$$

This matrix is not in row echelon form, because row 4 is a row whose entries are all zeros, but it is above row 5, which is a row that contains non-zero entries. So we need to move on to Step 4, where we will establish a pivot in row 4. So, we notice that column 6 is the first (and only) column to contain a non-zero entry below row 3. We will interchange rows 4 and 5 to put a non-zero entry in column 6 of row 4:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{array}\right] \quad R_4 \updownarrow R_5 \quad \left[\begin{array}{ccccc|c} 1 & 2 & 5 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We now have a matrix in row echelon form!

More on this example: During this example, I was focused on the matrix as an object, and ignoring the system of linear equations that generated it. But if we look at row 4 of our REF matrix, we see that this comes from the equation $0 = -12$. As this is never true, we know that our system of linear equations has no solutions. However, we first saw the equation $0 = -12$ in Step 3 of our row reduction, before we reached row echelon form. It will be important to distinguish between the instruction "Solve the system of linear equations" and "Obtain a row equivalent matrix in row echelon form." If our goal had been to solve the system, we could have stopped after step 3 and declared that there were no solutions. However, for this particular example, the goal was to put a matrix in row echelon form, so even after we saw that the corresponding system had no solutions, we needed to continue until we reached row echelon form.

Notes on combining elementary row operations: When I did Step 1 in my example, I painstakingly made sure that every row operation was performed separately. But in Step 2 I saved myself a bit of time and space by performing three elementary row operations "at the same time." I still wrote down the three calculations involved, but instead of changing the corresponding rows once per matrix, I jumped to the final matrix featuring three new rows. The rule to follow when deciding to do multiple operations at once is to make sure that you never change a row more than once, and you never use a row you are changing in an operation (3) for another row. So, for example, while I subtracted a (different) multiple of row 2 from three different rows in my Step 2, I did not alter row 2 at the same time. Other examples of things that would be fine would be to combine $R_1 \updownarrow R_2$ with $R_3 \updownarrow R_4$, but not $R_1 \updownarrow R_2$ with $R_3 \updownarrow R_2$, as the latter would lead to confusion as to which "R_2" you were referring to. Similarly, we can combine $2R_1$ with $2R_2$, but not $2R_1$ with $R_2 + 2R_1$, as this again leads to confusion as to which R_1 you are referring to. Note that we also do not want to combine steps $2R_2$ with $R_2 + 3R_1$ into a step like $2R_2 + 3R_1$. In this particular case, there wouldn't be any confusion, but later in the course it will be important to be able to single out our elementary row operations, and $2R_2 + 3R_1$, though it does result in a row equivalent matrix, is not an elementary row operation.