Lecture 2d

The Matrix Representation of a System of Linear Equations

(pages 69-73)

After you have spent a lot of time solving systems of equations using Gaussian elimination with back-substitution (which, presumably by now you haven't, but I assure you that *I* have), you soon come to the realization that the variables are simply placeholders. We never end up with an $x_1^2$ or $x_1x_2$. It's always coefficient times variable. Moreover, you never add the coefficient of an $x_1$ term to the coefficient of, for example, an $x_3$ term. So, if you do a good job of lining up your equations, you can completely ignore what variable you are dealing with, and instead simply make sure that keep your calculations in their own column. For example, the calculation

$$\begin{array}{rrcrcrcr} & -2x_1 & + & 3x_2 & - & 5x_3 & = & 7 \\ + & 2x_1 & + & x_2 & + & 3x_3 & = & -3 \\ \hline & & & 4x_2 & - & 2x_3 & = & 4 \end{array}$$

could be abbreviated as

$$\begin{array}{rrrrcr} & x_1 & x_2 & x_3 & & \\ \hline\hline & -2 & 3 & -5 & = & 7 \\ + & 2 & 1 & 3 & = & -3 \\ \hline & 0 & 4 & -2 & = & 4 \end{array}$$

And if you take for granted the fact that the first column will always correspond to $x_1$, etc., then you don't even need that row on the top. And so, from now on, we shall solve a system of linear equations by first writing all the coefficients in a rectangular array called a **matrix**. A general linear system of $m$ equations in $n$ unknowns will be represented by the matrix

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right]$$

You'll notice that we even further abbreviated by replacing all our equal signs with a vertical line. This is specifically known as the **augmented matrix** of the system. If we remove the vertical line and the final column, we end up with the following **coefficient matrix** of the system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

**Example:** The coefficient matrix for the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

is

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 4 \\ 2 & 5 & 1 \end{bmatrix}$$

and the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 3 & 4 & 11 \\ 2 & 5 & 1 & 3 \end{array}\right]$$

Now, as each row in the matrix corresponds to an equation in our system, the Gaussian elimination steps we just learned become the following:

<u>Definition</u> There are three types of **elementary row operations**, corresponding to the three steps of Gaussian elimination:

(1) Multiply one row by a non-zero constant.
(2) Interchange two rows.
(3) Add a multiple of one row to another row.

<u>Definition</u> The process of performing elementary row operations on a matrix is called **row reduction**.

<u>Definition</u> If the matrix $M$ is row reduced into a matrix $N$ by a sequence of elementary row operations, then we say that $M$ is **row equivalent** to $N$, and we write $M \sim N$. Note that this is the same as saying that the corresponding systems of equation are equivalent.

Course Author's Comments

I've got one more little thing to say in this lecture, and we won't really get into this idea of row reduction until the next lecture, but I feel that this is the point to let you know that matrices and row reduction are going to be really important concepts in this course. Really, REALLY, **REALLY** important. Elementary row operations are going to be as important to you as the other operations of adding, subtracting, multiplying and dividing. And we are going to do all sorts of things with these matrix things. So you should make sure that you take your time when you are working through Section 2.1 of the text, because I am not exaggerating when I say that you will not be able to pass this course if you can't do the questions from Section 2.1 quickly and confidently.

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Now, we've gone through a lot of trouble to abbreviate our work, so it would be a shame if we still had to go around saying "add -2 times row 2 to row 3", when something shorter like $R_3 - 2R_2$ would be shorter. More generally, we notate the elementary row operations as follows:

(1) Multiply row $i$ by a non-zero constant $c$ is notated as $cR_i$
(2) Interchange rows $i$ and $j$ is notated as $R_i \updownarrow R_j$.
(3) Add $c$ times row $i$ to row $j$ is notated as $R_j + cR_i$.

Notice that we switch the relative positioning of row $i$ and row $j$ in our notation for step 3. This is so that the first row we see lets us know which row is changing, as we switch from one matrix to the next.

Additionally, we will use the symbol $\sim$ (read "equivalent") to indicate that two matrices are row equivalent.

**Example:** In Lecture 2c, we went through the Gaussian elimination steps to find the general solution to the system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

Now, we will do the exact same steps, but using matrix notation:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ -1 & 3 & 4 & 11 \\ 2 & 5 & 1 & 3 \end{array}\right] R_2 + R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 5 & 3 & 7 \\ 2 & 5 & 1 & 3 \end{array}\right] R_3 - 2R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 1 & 3 & 11 \end{array}\right] R_2 - 5R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 0 & -12 & -48 \\ 0 & 1 & 3 & 11 \end{array}\right]$$