Lecture 2c

Gaussian Elimination (pages 64-68)

Now that we have an idea of our goal, we need to figure out how to manipulate our given equations in ways that will bring them closer to our goal, while not actually changing the solution.

Definition: We say that two systems of equations are **equivalent** if they have the same solution set.

So, we want to make sure that whatever steps we take to solve a system, we end up with an equivalent system. Gaussian elimination is performed with three such steps:

Step 1: Multiply one equation by a non-zero constant.

Example: The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ x_1 + 2x_2 - 3x_3 - 5x_4 &= 7 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned}$$

since we get the system on the right from the system on the left by multiplying the second equation by 1/2.

It is easy to see why this step results in an equivalent system, as we get that $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ is a solution to the equation $c(a_1x_1 + \cdots + a_nx_n) = cd$ if and only if it is also a solution to $a_1x_1 + \cdots + a_nx_n = d$, as we can simply multiply or divide by c. (Which is why we need $c \neq 0$.) So we are simply replacing one equation with an equivalent equation, and thus not changing the overall solution set.

Step 2: Interchange two equations. (That is, switch the positions of two equations.)

Example: The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \end{aligned}$$

since we get the system on the right from the system on the left by swapping the first and third equations.

I will take it as obvious why this step results in an equivalent system.

Step 3: Add a multiple of one equation to another equation.

Example: The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -9x_2 - 2x_3 - 6x_4 &= 39 \end{aligned}$$

since we get the system on the right from the system on the left by adding the second equation to the third.

It is a bit harder to see that this step results in an equivalent system. The first thing to realize is that we only need to show that the smaller system of the two involved equations is equivalent to the one we get by adding a multiple of one equation to the other. (Since any other equations in the system are unchanged, they will have the same effect on the overall solution set.) So, suppose we have the system

System 1

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= b \end{aligned}$$

If we multiply the first equation by the constant c and then add it to the second equation, we get

System 2

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ (ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n &= ca + b \end{aligned}$$

Our goal is to see that System 1 and System 2 have the same solution set. To do this, we will show that any solution of System 1 is a solution of System 2, and then showing that any solution to System 2 is a solution to System 1. So, let's assume that $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ is a solution to System 1. Then we can plug $\vec{s}$ in to our system to get the following equalities: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = a$ and $b_1s_1 + b_2s_2 + \cdots + b_ns_n = b$. Multiplying the first equality by c we get that $ca_1s_1 + ca_2s_2 + \cdots + ca_ns_n = ca$, and then adding this to the second equality gives us that $ca_1s_1 + ca_2s_2 + \cdots + ca_ns_n + b_1s_1 + b_2s_2 + \cdots + b_ns_n = ca + b$. Rearranging, we see that $(ca_1 + b_1)s_1 + (ca_2 + b_2)s_2 + \cdots + (ca_n + b_n)s_n = ca + b$, and so $\vec{s}$ is a solution of both $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ and $(ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n = ca + b$. That is, $\vec{s}$ is a solution to System 2.

With the first part of our proof complete, lets now assume that $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ is a solution of System 2. Then we get the following equalities: $a_1t_1 + a_2t_2 + \cdots + a_nt_n = a$ and $(ca_1 + b_1)t_1 + (ca_2 + b_2)t_2 + \cdots + (ca_n + b_n)t_n = ca + b$. Multiplying the first equality by c gives us $ca_1t_1 + ca_2t_2 + \cdots + ca_nt_n = ca$. Subtracting this from $(ca_1+b_1)t_1+(ca_2+b_2)t_2+\cdots+(ca_n+b_n)t_n = ca+b$, we get $b_1t_1+b_2t_2+\cdots+b_nt_n = b$. So $\vec{t}$ is a solution of both $a_1x_1+a_2x_2+\cdots+a_nx_n = a$ and $b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$. That is, $\vec{t}$ is a solution to System 1.

Now that we are sure that our steps will result in an equivalent system, we come to the important part of USING these steps to solve a system. The example I used for Step 3, as well as being an example of Step 3, also demonstrates why this technique is called the "elimination" method, as we eliminated the variable x_1 from the third equation. Recall that our ideal outcome would be to have a list of equations, each of which only involves one variable. So our goal is to use these steps to eliminate as many variables as possible from each equation. We'll discuss how we go about doing this in a the coming lectures, but for now lets just focus on the mechanics of performing these steps. To that end, let's look at some examples.

Example: Find the solution set for the following system of equations:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

Now, our first goal is going to be to eliminate the variable x_1 from two of the three equations. We can eliminate x_1 from the second equation by adding the first equation to the second equation, giving us the equivalent system:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

We can remove the variable x_1 from the third equation by adding -2 times the first equation to the third equation. I find that students frequently want to do such calculations in their head. I also find that students frequently make mistakes in these calculations! Therefore, I strongly recommend that you write out this calculation, something like this:

$$\begin{array}{rcrcrcr} -2(x_1 & + & 2x_2 & - & x_3 & = & -4) \\ \hline -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \\ \hline & & x_2 & + & 3x_3 & = & 11 \end{array}$$

So our new system is

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

This completes our first goal. Our second goal is to remove the x_2 variable from one of the equations that already has the x_1 variable removed, leaving only the x_3 variable. I will go about this by adding -5 times the third equation to the second equation. The calculation for this is

$$\begin{array}{rcrcr} -5(x_2 & + & 3x_3 & = & 11) \\ \hline -5x_2 & - & 15x_3 & = & -55 \\ 5x_2 & + & 3x_3 & = & 7 \\ \hline & - & 12x_3 & = & -48 \end{array}$$

So our new system is

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & & - & 12x_3 & = & -48 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

In the previous lecture, I did the back-substitution steps to get that the general solution is the single vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

Example: Find the solution set for the system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13 \\ 2x_1 & + & 4x_2 & - & 4x_3 & + & 8x_4 & = & 26 \\ 2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & + & 18 \end{array}$$

As in the previous example, our first goal will be to eliminate the x_1 variable from two of the three equations. I can do this by adding -2 times the first equation to each of the second and third equations. The calculations are:

$$\begin{array}{rcrcrcrcr} -2(x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13) \\ \hline -2x_1 & - & 4x_2 & + & 4x_3 & - & 8x_4 & = & -26 \\ 2x_1 & + & 4x_2 & - & 4x_3 & + & 8x_4 & = & 26 \\ \hline & & & & & & 0 & = & 0 \end{array}$$

$$\begin{array}{rcrcrcrcr}
-2(x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13) \\
\hline
-2x_1 & - & 4x_2 & + & 4x_3 & - & 8x_4 & = & -26 \\
2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & = & 18 \\
\hline
 & & & & x_3 & - & 3x_4 & = & -8
\end{array}$$

So our new system is

$$\begin{array}{rcrcrcrcr}
x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13 \\
 & & & & & & 0 & = & 0 \\
 & & & & x_3 & - & 3x_4 & = & -8
\end{array}$$

This example illustrates two important things that can happen when you are solving a system of equations. The first is the fact that you may end up eliminating an entire equation, as happened with our second equation. This does not mean that you have made a mistake! It simply means that the solutions of that equation are already solutions of the other equations, so the eliminated equation wasn't providing us any additional information. Technically, we do not need even need the placeholder $0 = 0$ equation, but we will be looking at a slight variation of elimination in the next lecture where we will want to keep such a placeholder, so I will leave it in for now too.

The second thing that happened is that in the process of eliminating x_1 from two of the three equations, we also eliminated x_2. Normally, after we eliminate x_1 from all but one equation, we would work to eliminate x_2 from all but one of the remaining equations. However, since it doesn't appear in *any* of the remaining equations, we would simply consider this task done, and move on to removing x_3 from all but one of the remaining equations (where "remaining equations" refers to the same equations as if we were considering x_2). I'll go into this more in the next lectures, when we use our variation of elimination, so all that is important for now is the fact that sometimes you eliminate variables even when you aren't trying to.

Again, I did the back-substitution steps for this system in the previous lecture, and we found that the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -8 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Example: Find the general solution to the system

$$\begin{array}{rcrcrcr}
3x_1 & + & 5x_2 & + & x_3 & = & -8 \\
x_1 & + & x_2 & - & x_3 & = & -2 \\
-x_1 & + & 2x_2 & + & 7x_3 & = & 7
\end{array}$$

The first thing we want to do is eliminate x_1 from two of our equations. Instead of eliminating x_1 from the second and third equations as we have in the past, we can notice that the second equation has a starts with a pleasant x_1 term, so our life will be easier if we use this equation to eliminate x_1 from the first and third equations. In fact, why don't we go ahead and interchange equations one and two, giving us the equivalent system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ 3x_1 & + & 5x_2 & + & x_3 & = & -8 \\ -x_1 & + & 2x_2 & + & 7x_3 & = & 7 \end{array}$$

Now we can add -3 times the first equation to the second equation, and add the first equation to the third equation, giving us the equivalent system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & 2x_2 & + & 4x_3 & = & -2 \\ & & 3x_2 & + & 6x_3 & = & 5 \end{array}$$

We see now that x_2 will be a leading variable, so now our goal is to eliminate x_2 from either the second or third equation. This time I notice that all the coefficients in the second equation are a multiple of 2, so we can easily multiply equation two by 1/2 to get the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & x_2 & + & 2x_3 & = & -1 \\ & & 3x_2 & + & 6x_3 & = & 5 \end{array}$$

Now we can easily add -3 times equation two to equation three, getting

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & x_2 & + & 2x_3 & = & -1 \\ & & & & 0 & = & 8 \end{array}$$

First of all, yes, I could have skipped the "multiply the second equation by 1/2" step and instead added -3/2 times equation two to equation three. But with bigger systems we often find value in having our leading variables have the coefficient "1" with them. However, I should not distract you from the fact that we end up with 0=8 as our third equation, which is a contradiction. As such, we see that our system has no solutions.