Lecture 2b

Back-Substitution

(pages 64-68)

The first method we will use to solve a system of linear equations is known as Gaussian elimination with back-substitution. The idea is that we are hoping for a solution of the form

$$x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$$

so we want to manipulate our given equations in ways that will bring them closer to this simple form, while not actually changing the solution. But before we go into the details of how we manipulate our equations, lets take a look at what sorts of results we might end up with. Because a list like

$$x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n$$

would correspond to a solution set with exactly one element, but we will sometimes have an infinite number of elements, and sometimes none. So before we start trying to alter our given system, we should have an idea of what the final result will look like. To that end, let us look at some examples:

Example: Find the solution set for the following system of equations:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & & - & 12x_3 & = & -48 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

This is an example of a system where we are done with our elimination steps, so we can now finish finding the solution set for the system using **back-substitution**. That is, we will start with the fact that we have found a value for x_3, since our second equation ($-12x_3 = -48$) tells us that $x_3 = 4$. Now, we *substitute* the value $x_3 = 4$ *back* into the other equations, giving us the new system

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & 4 & = & -4 \\ & & & & x_3 & = & 4 \\ & & x_2 & + & 12 & = & 11 \end{array}$$

Subtracting 12 from both sides of our third equation tells us that $x_2 = 11 - 12 = -1$, so we can substitute $x_2 = -1$ into the first equation. We get $x_1 + 2(-1) - 4 = -4$, so $x_1 = 2$. Thus, we have seen that the only possible solution to the system is

$$x_1 = 2,\ x_2 = -1,\ x_3 = 4$$

That is, the solution set of our system is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$. So this is one of our "ideal" examples, where we end up with each variable equal to a constant. It is an excellent way to see where the term back-substitution comes from, but now we will look at the other types of solutions sets.

Example: Find the solution set for the following system:

$$x_1 + x_2 = 1$$

Yes, a "system" can consist of only one equation! You have probably encountered questions such as this one before. It could be rephrased as "find me two numbers that add up to 1". Of course this is going to have multiple answers! The first that jump to mind are 0 and 1, or alternatively 1 and 0. 1/2 and 1/2 will do quite nicely too, as well as 2 and -1. Pretty quickly you figure out that you can start with any number, and then simply subtract that number from 1 to get the second number you need. That is, we end up defining our value for x_1 in terms of a *chosen* value for x_2. The way we write this down is by the use of a **parameter**. So, we replace the *variable* x_2 with a *parameter* s in $\mathbb{R}$. While s will be allowed to vary through all the elements of $\mathbb{R}$, for the purposes of finding the solution set of the system, parameters are considered to be constants. So our system turns into $x_1 + s = 1$, so $x_1 = 1 - s$. Thus, the solution to our system is $x_1 = 1 - s$, $x_2 = s$, or $\left\{ \begin{bmatrix} 1-s \\ s \end{bmatrix} : s \in \mathbb{R} \right\}$. Since we have a unique answer for every choice of s from $\mathbb{R}$, we see that there are an infinite number of solutions to the system $x_1 + x_2 = 1$.

Example: Find the solution set for the system

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13 \\ & & & & x_3 & - & 3x_4 & = & -8 \end{array}$$

This example is very much like the previous one, in that we will need to introduce parameters to find the solution set. Parameters will always be necessary when there are more variables than equations! (Unless there are no solutions, but we'll look at an example of that next.) So how do we decide which variables to replace with parameters? First we identify the **leading variables** in the system. These are the variables that appear first in one of the equations. In this example, the leading variable in the first equation is x_1, and the leading variable in the second equation is x_3 (as the second equation does not have an

x_1 or x_2 term). The leading variables are the ones that we will end up solving for, and NOT the ones that we replace with parameters. Instead, all variables that are not leading variables get turned into parameters. For this example, that means that we replace the variables x_2 and x_4 with parameters, say s and t. This turns our system into

$$\begin{array}{ccccccccc} x_1 & + & 2s & - & 2x_3 & + & 4t & = & 13 \\ & & & & x_3 & - & 3t & = & -8 \end{array}$$

At this point, we are ready to start the back-substitution process. First, we use our second equation to find that $x_3 = -8 + 3t$. Then we substitute this back into the first equation to get $x_1 + 2s - 2(-8 + 3t) + 4t = 13$. Solving for x_1, we see that $x_1 = -3 - 2s + 2t$. Thus, our solution set is

$$\left\{ \begin{bmatrix} -3 - 2s + 2t \\ s \\ -8 + 3t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

That's not how we usually write our solutions, though. Instead, we drop the set notation, and separate out the parameters, writing our solution set as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -8 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

In doing so, we emphasize the geometrical interpretation of the solution set. For example, in this case we see that the solution set defines a plane in $\mathbb{R}^4$. This format will be known as the **standard form** for the general solution of a system of linear equation, and is the format you are expected to use when presenting your answers.

Example: Find the solution set for the system

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & -1 \end{array}$$

In the previous lecture, I used this system as an example of a system that has no solutions. Of interest now is what this system would look like at the end of the Gaussian elimination process. Well, it would look like this:

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ & & 0 & = & -2 \end{array}$$

The "0=-2" equation is the one of interest at this time, as it is the contradiction to the assumption that the system had solutions. Any system that has no solutions will generate an equation of the form "$0 = c$" for some $c \neq 0$ during the Gaussian elimination process, and as soon as such an equation appears we immediately know that the system has no solutions.