Lecture 2a

Systems of Linear Equations

(pages 63-64)

In Lecture 1s, we needed to find a point on two planes. To do this, we needed to find a point (x_1, x_2, x_3) that was a solution to two equations for planes. This is one of many examples of a system of linear equations.

At this point, it may seem odd that the equation for a plane in $\mathbb{R}^3$, or more generally for a *hyperplane*, is being referred to as a *linear* equation. As you will see in the definition of a linear equation (coming next!), an equation is considered linear if it doesn't involve multiplying variables. For example, $2x + 3y = 5$ is a linear equation, but $2xy = 5$, $2x^2 + 3y = 5$, and $2x + 3\sqrt{y} = 5$ are NOT linear equations. Our use of the word "linear" in this course refers to linear combinations, not lines.

Definition: A **linear equation** in n variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

The numbers $a_1, \ldots, a_n$ are called the **coefficients** of the equation, and b is usually referred to as the "right-hand side," or the "constant term." The x_i are the unknowns or variables to be solved for.

Definition: A vector $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ in $\mathbb{R}^n$ is called a **solution** of a linear equation if the equation is satisfied when we make the substitution $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$.

Example: $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ is a solution to $4x_1 + 3x_2 + 2x_3 = 9$ because $4(2) + 3(-3) + 2(5) = 9$.

Definition: A general **system of m linear equations in n variables** is written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Note that the coefficient a_{ij} is in the i-th equation, with the j-th variable. I usually remember this by remembering that the last coefficient is a_{mn}, not a_{nm}, so I know that number representing the equation comes first, and the number representing the variable comes second.

Definition: The **solution set** to a system of linear equations is the collection of all vectors that are solutions to all the equations in the system. This set will be a subset of $\mathbb{R}^n$, but it may be the empty set. (That is, there may not be any vectors that are solutions to all the equations.)

Example: The solution set for the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= -1 \end{aligned}$$

is $\emptyset$, since we can never simultaneously have $x_1 + x_2 = 1$ AND $x_1 + x_2 = -1$.

Example: The solution set for the system

$$\begin{aligned} x_1 &= 1 \\ x_1 + x_2 &= 3 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

In general, it is not this easy to find the solution set for a system of equations, but it is always possible. Our goal for this week is to develop an algorithm for finding solution sets.