# Solution to Practice 1n

**A1(a)** First I calculate $\vec{v}\cdot\vec{u} = (0)(3)+(1)(-5) = -5$ and $||\vec{v}||^2 = 0^2+1^2 = 1$. So $\text{proj}_{\vec{v}}\vec{u} = (-5/1)\vec{v} = -5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$. This means that $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Double checking, we verify that $\text{proj}_{\vec{v}}\vec{u}+\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \vec{u}$, and that $\vec{v}\cdot\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 0 \end{bmatrix} = (0)(3)+(1)(0) = 0$.

**A1(c)** First I calculate $\vec{v}\cdot\vec{u} = (0)(-3)+(1)(5)+(0)(2) = 5$ and $||\vec{v}||^2 = 0^2+1^2+0^2 = 1$. So $\text{proj}_{\vec{v}}\vec{u} = (5/1)\vec{v} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$. This means that $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$. Double checking, we verify that $\text{proj}_{\vec{v}}\vec{u}+\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \vec{u}$, and that $\vec{v}\cdot\text{perp}_{\vec{v}}\vec{u} = (0)(-3)+(1)(0)+(0)(2) = 0$.

**A1(e)** First I calculate $\vec{v}\cdot\vec{u} = (1)(-1)+(1)(-1)+(0)(2)+(-2)(-1) = 0$. This means that $\vec{u}$ is orthogonal to $\vec{v}$, and thus we know immediately that $\text{proj}_{\vec{v}}\vec{u} = \vec{0}$ and $\text{perp}_{\vec{v}}\vec{u} = \vec{u}$. As such, we easily see that $\text{proj}_{\vec{v}}\vec{u}+\text{perp}_{\vec{v}}\vec{u} = \vec{0}+\vec{u} = \vec{u}$, and that $\vec{v}\cdot\text{perp}_{\vec{v}}\vec{u} = \vec{v}\cdot\vec{u} = 0$.

**A2(a)** First I calculate $\vec{v}\cdot\vec{u} = (1)(3)+(1)(-3) = 0$. This means that $\vec{u}$ is orthogonal to $\vec{v}$, so $\text{proj}_{\vec{v}}\vec{u} = \vec{0}$ and $\text{perp}_{\vec{v}}\vec{u} = \vec{u} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

**A2(b)** First I calculate $\vec{v}\cdot\vec{u} = (2)(4)+(3)(-1)+(-2)(3) = -1$ and $||\vec{v}||^2 = 2^2+3^2+(-2)^2 = 17$. Then $\text{proj}_{\vec{v}}\vec{u} = (-1/17)\vec{v} = \begin{bmatrix} -2/17 \\ -3/17 \\ 2/17 \end{bmatrix}$, and $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/17 \\ -3/17 \\ 2/17 \end{bmatrix} = \begin{bmatrix} 70/17 \\ -14/17 \\ 49/17 \end{bmatrix}$.

**A2(c)** First I calculate $\vec{v}\cdot\vec{u} = (-2)(5)+(1)(-1)+(-1)(3) = -14$ and $||\vec{v}||^2(-2)^2+1^2+(-1)^2 = 6$. Then $\text{proj}_{\vec{v}}\vec{u} = (-14/6)\vec{v} = \begin{bmatrix} 14/3 \\ -7/3 \\ 7/3 \end{bmatrix}$, and $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 14/3 \\ -7/3 \\ 7/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 4/3 \\ 2/3 \end{bmatrix}$.

**A2(d)** First I calculate $\vec{v}\cdot\vec{u} = (1)(4) + (1)(1) + (-2)(-2) = 9$ and $||\vec{v}||^2 = 1^2+1^2+(-2)^2 = 6$. Then $\text{proj}_{\vec{v}}\vec{u} = (9/6)\vec{v} = \begin{bmatrix} 3/2 \\ 3/2 \\ -3 \end{bmatrix}$, and $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ -3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \\ 1 \end{bmatrix}$.

**A2(e)** First I calculate $\vec{v}\cdot\vec{u} = (-1)(2) + (2)(-1) + (1)(2) + (-3)(1) = -5$ and $||\vec{v}||^2 = (-1)^2 + 2^2 + 1^2 + (-3)^2 = 15$. Then $\text{proj}_{\vec{v}}\vec{u} = (-5/15)\vec{v} = \begin{bmatrix} 1/3 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix}$, and $\text{perp}_{\vec{v}}\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -2/3 \\ -1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 7/3 \\ 0 \end{bmatrix}$.

**D5(a)** First I calculate $\vec{u}\cdot\vec{x} = (1)(2)+(1)(5)+(-1)(3) = 4$ and $||\vec{u}||^2 = 1^2+1^2+(-1)^2 = 3$. then $\text{proj}_{\vec{u}}\vec{x} = (4/3)\vec{u} = \begin{bmatrix} 4/3 \\ 4/3 \\ -4/3 \end{bmatrix}$. Thus, $\text{perp}_{\vec{u}}\vec{x} = \vec{x} - \text{proj}_{\vec{u}}\vec{x} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ -4/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 11/3 \\ 13/3 \end{bmatrix}$. Now we calculate $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 11/3 \\ 13/3 \end{bmatrix} = 2/3 + 11/3 - 13/3 = 0$. As such, $\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}\vec{x}) = 0$.

**D5(b)** We have $\text{perp}_{\vec{u}}\vec{x} = \vec{x} - \text{proj}_{\vec{u}}\vec{x} = \vec{x} - \frac{\vec{u}\cdot\vec{x}}{||\vec{u}||^2}\vec{u}$. Thus,

$$\begin{aligned}
\text{proj}_{\vec{u}}(\text{perp}_{\vec{u}}\vec{x}) &= \frac{\vec{u}\cdot(\vec{x} - \frac{\vec{u}\cdot\vec{x}}{||\vec{u}||^2}\vec{u})}{||\vec{u}||^2}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{x} - \frac{\vec{u}\cdot\vec{x}}{||\vec{u}||^2}\vec{u}\cdot\vec{u}}{||\vec{u}||^2}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{x} - \frac{\vec{u}\cdot\vec{x}}{||\vec{u}||^2}||\vec{u}||^2}{||\vec{u}||^2}\vec{u} \\
&= \frac{\vec{u}\cdot\vec{x} - \vec{u}\cdot\vec{x}}{||\vec{u}||^2}\vec{u} \\
&= 0\vec{u} \\
&= \vec{0}
\end{aligned}$$

**D5(c)** Geometrically, $\text{perp}_{\vec{u}}\vec{x}$ is the part of $x$ that does not have anything in common with $\vec{u}$. That is, it is the part of $\vec{x}$ that is perpendicular to $\vec{u}$. As such, when we take the projection of this onto $\vec{u}$, as it had nothing in common with $\vec{u}$, we are left with the zero vector.