Lecture 1p

Cross Product

(pages 50-52)

Back in Lecture 1k (or Section 1.3) we defined the scalar product of two vectors. At this point, we will look at another type of vector "product"–the cross product.

Definition: The **cross-product** of vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is defined by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Note that the cross product is only defined in $\mathbb{R}^3$!

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} (2)(6) - (3)(-5) \\ (3)(4) - (1)(6) \\ (1)(-5) - (2)(4) \end{bmatrix} = \begin{bmatrix} 27 \\ 6 \\ -13 \end{bmatrix}$

Now, the formula for the cross product can be difficult to remember, but I find it the easiest way. The textbook discuss a method for calculating the cross product, and I will present my own slightly different method that I feel is at least a bit easier. As in the text book, we begin listing the components of $\vec{u}$ and $\vec{v}$, one on top of the other:

$$\begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}$$

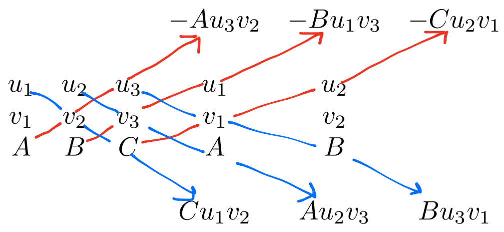
Next, I add the letters A, B and C beneath these component:

$$\begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ A & B & C \end{matrix}$$

Next, I duplicate the first two columns on the right:

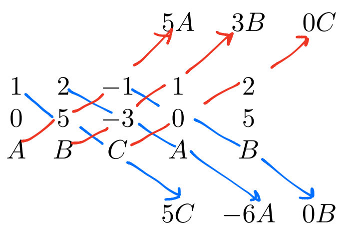
$$\begin{matrix} u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ A & B & C & A & B \end{matrix}$$

Now, I look at the diagonal products that contain three values, giving a positive value to products from top-left to bottom-right, and a negative value to products from bottom-left to top-right:



Then if we combine the A terms, we get $-Au_3v_2 + Au_2v_3 = A(u_2v_3 - u_3v_2)$, so removing the A we have the first component of $\vec{u}\times\vec{v}$. Similarly, $-Bu_1v_3+Bu_3v_1$ leads us to the second component $u_3v_1 - u_1v_3$, and $-Cu_2v_1 + Cu_1v_2$ leads us to the third component $u_1v_2 - u_2v_1$. So the added A, B, and C simply serve as placeholders to let us know which component of $\vec{u} \times \vec{v}$ we are calculating.

Example: Calculate the cross product of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}$



So our A terms are $5-6=-1$, our B terms are $3+0=3$, and our C terms are $0+5=5$. Thus, $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$.

Again, I recommend simply memorizing the defining formula. By the time you finish doing a large number of cross product questions, you will probably have the formula memorized. I will use my technique in the solutions to the practice problems for this lecture, but in the rest of the course I'm simply going to use the formula. Of course, you should use whichever technique you like best. But before we move on to using the cross product to calculate all sorts of interesting things, we will take note of some "useful properties" of the cross product:

Theorem 1.5.1 For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

(1) $\vec{x} \times \vec{y} = -y \times \vec{x}$
(2) $\vec{x} \times \vec{x} = \vec{0}$
(3) $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$
(4) $(t\vec{x}) \times \vec{y} = t(\vec{x} \times \vec{y}) = \vec{x} \times (t\vec{y})$

Course author's note: The final $= \vec{x} \times (t\vec{y})$ of line (4) does not appear in the textbook, but is true and useful nevertheless, so I choose to add it. Feel free to use it!

I shall provide a proof of line (1) as a sample of how to prove the theorem:

Proof of (1): Suppose $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then

$$\begin{aligned} -\vec{y} \times \vec{x} &= -\begin{bmatrix} y_2x_3 - y_3x_2 \\ y_3x_1 - y_1x_3 \\ y_1x_2 - y_2x_1 \end{bmatrix} \\ &= \begin{bmatrix} y_3x_2 - y_2x_3 \\ y_1x_3 - y_3x_1 \\ y_2x_1 - y_1x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \\ &= \vec{x} \times \vec{y} \end{aligned}$$

I find that students often want to give "theoretical" proofs, such as proving line (1) by saying "well, we just switch the order, so that puts the negative in the wrong place, so we pull out a negative". I consider that to be more of a sketch of a proof than an actual proof, and as an instructor sometimes that is all I give, if I feel that the details of the proof don't enhance the learning experience. However, at this stage in your studies, I recommended writing out all the details of the proof as I did above. I find that actually writing out all the terms and moving them around will give you a better understanding of the underlying features of the calculation you are performing. I will also point out, therefore, that the author of the textbook decided to leave the proof of Theorem 1.5.1 "to the reader", not because the proof was unimportant, but because the act of going through the proof has educational value. Especially to students who struggle with proofs.

I am also frequently asked, fearfully, whether or not there will be proofs on the final exam. I suspect some of the fear comes from the fact that students often overlook the many easy proofs such as the various "Theorem 1" we have already encountered. Other students seem put off by the easiness of such theorems, and seem not to know where to begin when the path seems so clear. Mostly, I take this question as a lack of understanding about math classes. EVERY QUESTION IS A PROOF! If I ask you to calculate a cross product, I do not expect you to simply give me an answer. Instead, I expect you to demonstrate some knowledge of the definition of the cross product (using either the textbook technique or my technique for setting up the cross product counts), and display the necessary calculations to get the answer. So, you use a combination of definitions and/or theorems, and maybe some calculations, to present a result. Sounds like a proof to me!