Lecture 1o

Minimum Distance

(pages 44-47)

Now that we are comfortable computing proj and perp, we can return to the issue of finding the distance from a point to a line.

<u>Definition</u> The **distance from the line** $\vec{x} = \vec{p} + t\vec{d}$ **to the point** Q is the minimum distance from the point Q to any point on the line, which equals $||\text{perp}_{\vec{d}}\vec{PQ}||$.

Moreover, we see that if we want to calculate the point R that is the point closest to the line, we can either start at the point P and add the vector $\text{proj}_{\vec{d}}\vec{PQ}$, or we can start at the point Q and SUBTRACT the vector $\text{perp}_{\vec{d}}\vec{PQ}$. (Note: we need to subtract the vector, since, as we see in the drawing, the vector $\text{perp}_{\vec{d}}\vec{PQ}$ is pointing in the opposite direction from what we want.)

<u>Summarizing</u> Let R be the point on the line $\vec{x} = \vec{p} + t\vec{d}$ that is closest to the point Q. Then

$$\vec{r} = \vec{p} + \text{proj}_{\vec{d}}\vec{PQ} = \vec{q} - \text{perp}_{\vec{d}}\vec{PQ}$$

Note that although our drawing takes place in $\mathbb{R}^2$, our results are still valid in the general $\mathbb{R}^n$.

Example: Find the point on the line $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ that is closest to the point $Q(1, 2, 1)$, and calculate the distance from Q to the line.

First we note that P is $(0, 2, -2)$, and then we calculate that

$$\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Next, to find $\text{proj}_{\vec{d}} \vec{PQ}$, we calculate

$\vec{d} \cdot \vec{PQ} = (3)(1) + (0)(0) + (-3)(3) = -6$ and $||\vec{d}||^2 = (3)^2 + (0)^2 + (-3)^2 = 18$

This means that

$$\text{proj}_{\vec{d}} \vec{PQ} = \frac{\vec{d} \cdot \vec{PQ}}{||\vec{d}||^2} \vec{d} = \frac{-6}{18} \vec{d} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{r} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, and thus the point on the line closest to Q is $R(-1, 2, -1)$. Continuing on, we get

$$\text{perp}_{\vec{d}} \vec{PQ} = \vec{PQ} - \text{proj}_{\vec{d}} \vec{PQ} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

And so the distance from Q to the line is $||\text{perp}_{\vec{d}} \vec{PQ}|| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.

We can also look for the distance (that is, the minimum distance) from a point to a plane in $\mathbb{R}^3$. The advantage here is that we already know of one vector that is perpendicular to our plane–the normal vector. So in this case we can compute the distance by looking at the PROJECTION of $\vec{PQ}$ onto the normal vector $\vec{n}$. And we can also use $\text{proj}_{\vec{n}} \vec{PQ}$ to find the point R on the plane closest to Q, although because our vector $\text{proj}_{\vec{n}} \vec{PQ}$ points out from the plane, we will need to multiply it by -1 to reverse its direction.

Summarizing Let R be the point on the plane $ax_1 + bx_2 + cx_3 = d$ closest to the point Q, and let P be any point on the plane and $\vec{n}$ be the normal vector for the plane. Then

$$\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \vec{p} + \text{perp}_{\vec{n}}\vec{PQ}$$

and the distance from Q to the plane is

$$||\text{proj}_{\vec{n}}\vec{PQ}||$$

In the text, the author writes $\vec{OR} = \vec{OQ} + \text{proj}_{\vec{n}}\vec{QP}$, while I use the equation $\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ}$. The change from $\vec{OR}$ to $\vec{r}$ and $\vec{OQ}$ to $\vec{q}$ is common, but I wish to further point out that $\text{proj}_{\vec{n}}\vec{QP} = \text{proj}_{\vec{n}}(-\vec{PQ}) = -\text{proj}_{\vec{n}}\vec{PQ}$ (by the linearity properties of proj). So the two equations really are the same.

Example: Find the distance from the point $Q(1, 2, 1)$ to the plane $2x_1 + 3x_2 - x_3 = 4$, and find the point R on the plane closest to Q.

To do this, we first want to find a point P on the plane. If we set $x_2 = x_3 = 0$, we get $2x_1 = 4 \Rightarrow x_1 = 2$. So we see that $P(2, 0, 0)$ is a point on our plane. This gives us $\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. We can read off from the equation that $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. So, the next step is to find $\text{proj}_{\vec{n}}\vec{PQ}$, which we begin by calculating

$$\vec{n} \cdot \vec{PQ} = (2)(-1) + (3)(2) + (-1)(1) = 3 \text{ and } ||\vec{n}||^2 = 2^2 + 3^2 + (-1)^2 = 14$$

So

$$\text{proj}_{\vec{n}}\vec{PQ} = \frac{\vec{n}\cdot\vec{PQ}}{||\vec{n}||^2}\vec{n} = \frac{3}{14}\vec{n} = \begin{bmatrix} 6/14 \\ 9/14 \\ -3/14 \end{bmatrix}$$

and thus the distance from Q to the plane is $\sqrt{(6/14)^2 + (9/14)^2 + (-3/14)^2} = 3\sqrt{14}$. Now, to find the point R, we first need to compute

$$\vec{r} = \vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6/14 \\ 9/14 \\ -3/14 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 19/14 \\ 17/14 \end{bmatrix}$$

So the point on the plane closest to Q is $R(4/7, 19/14, 17/14)$. One can double check this answer by plugging R into the equation of the plane and verifying that R is in fact a point on the plane. It doesn't guarantee that you've found the point on the plane closest to Q, but the odds that a calculation error would result in a point on the plane are small.

And, of course, all our results for the plane in $\mathbb{R}^3$ generalize as results for hyperplanes in $\mathbb{R}^n$.

Example: Find the point on the hyperplane $x_1+x_2-x_3-x_4 = 2$ that is closest to the point $Q(0,0,0,0)$, and calculate the distance from Q to the hyperplane.

First, we need to find a point P on the hyperplane. By inspection, we notice that $1+1-0-0 = 2$, so $(1,1,0,0)$ is a point on the plane. And since Q is $(0,0,0,0)$, we have that $\vec{PQ} = -\vec{p} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. We can read off $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ from the equation of our hyperplane, and thus $\vec{n}\cdot\vec{PQ} = (1)(-1) + (1)(-1) + (-1)(0) + (-1)(0) = -2$. Next, we compute $||\vec{n}||^2 = 1^2+1^2+(-1)^2+(-1)^2 = 4$. So $\text{proj}_{\vec{n}}\vec{PQ} = (-2/4)\vec{n} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. As such, the point on the plane closest to Q is $\vec{q} - \text{proj}_{\vec{n}}\vec{PQ} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$, or in point form, $(1/2, 1/2, -1/2, -1/2)$. And the distance from Q to the plane is $||\text{proj}_{\vec{n}}\vec{PQ}|| = \sqrt{(-1/2)^+(-1/2)^2 + (1/2)^2 + (1/2)^2} = 1$.