Solution to Practice 1l

A2(a) Let $\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Then $||\vec{x}|| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$. So a unit vector in the direction of $\vec{x}$ is $\frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$

A2(c) Let $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$. Then $||\vec{y}|| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5}$. So a unit vector in the direction of $\vec{y}$ is $\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$

A2(e) Let $\vec{z} = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. Then $||\vec{z}|| = \sqrt{(-2)^2 + (-2)^2 + 1^2 + 0^2} = \sqrt{9} = 3$. So a unit vector in the direction of $\vec{z}$ is $\frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{bmatrix}$

A4(a) First we compute $||\vec{x}|| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$ and $||\vec{y}|| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$. Now, for the triangle inequality, we compute $\vec{x} + \vec{y} = \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}$, which gives us $||\vec{x} + \vec{y}|| = \sqrt{6^2 + 4^2 + 6^2} = \sqrt{88}$. Since $\sqrt{26} > \sqrt{25} = 5$, $\sqrt{30} > \sqrt{25} = 5$, and $\sqrt{88} < \sqrt{100} = 10$, We see that $||\vec{x} + \vec{y}|| < 10 = 5 + 5 < ||\vec{x}||| + ||\vec{y}||$, so the triangle inequality holds. To show the Cauchy-Schwartz inequality, we need to compute $\vec{x} \cdot \vec{y} = (4)(2) + (3)(1) + (1)(5) = 16$, and we see that $||\vec{x}||\ ||\vec{y}|| = \sqrt{26}\sqrt{30} = \sqrt{780} > \sqrt{256} = 16 = |\vec{x} \cdot \vec{y}|$, as desired.

A4(b) First we compute $||\vec{x}|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$ and $||\vec{y}|| = \sqrt{(-3)^2 + 2^2 + 4^2} = \sqrt{29}$. Now, for the triangle inequality, we compute $\vec{x} + \vec{y} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$, which gives us $||\vec{x} + \vec{y}|| = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41}$. Since $\sqrt{6} > \sqrt{4} = 2$, $\sqrt{29} > \sqrt{25} = 5$, and $\sqrt{41} < \sqrt{49} = 7$, we have $||\vec{x} + \vec{y}|| < 7 = 2 + 5 < ||\vec{x}|| + ||\vec{y}||$, so the triangle inequality holds. To show the Cauchy-Schwarz inequality, we need to compute $\vec{x} \cdot \vec{y} = (1)(-3) + (-1)(2) + (2)(4) = 3$, and we see that $||\vec{x}||\ ||\vec{y}|| = \sqrt{6}\sqrt{29} = \sqrt{174} > \sqrt{9} = 3 = |\vec{x} \cdot \vec{y}|$, as desired.

A5(b) $\begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = (-3)(2) + (1)(-1) + (7)(1) = 0$, so the vectors ARE orthogonal.

A5(d) $\begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 3 \\ 0 \end{bmatrix} = (4)(-1) + (1)(4) + (0)(3) + (-2)(0) = 0$, so the vectors ARE orthogonal.

A5(f) $\begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \\ 1 \end{bmatrix} = (1/3)(3/2)+(2/3)(0)+(-1/3)(-3/2)+(3)(1) = 1/20 + 1/2 + 3 = 4$, so the vectors ARE NOT orthogonal.

A6(a) First we compute $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ k \end{bmatrix} = (3)(2) + (-1)(k) = 6 - k$. So the vectors are orthogonal if and only if $6 - k = 0$, that is if and only if $k = 6$.

A6(d) First we compute $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} k \\ k \\ -k \\ 0 \end{bmatrix} = k + 2k - 3k + 0 = 0$. So the vectors are orthogonal for all values of k.

A11(a) Well, anyone who reads part (b) before doing part (a) will hopefully realize that setting $\vec{u} = \vec{0}$ makes this equation true for any values of $\vec{v}$ and $\vec{w}$. So, for example, we can use $\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ as a counterexample. Thus, the statement is FALSE.

A11(b) Actually, even if $\vec{u} \neq \vec{0}$, this equation still isn't true. Simply consider the fact that if $\vec{u}$ is orthogonal to $\vec{v}$, then $\vec{u}$ is orthogonal to all the scalar multiples of $\vec{v}$. (You can deduce this pretty easily from line 4 of theorem 1.) So, I can pick $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and we have $\vec{u} \cdot \vec{v} = 0 = \vec{u} \cdot \vec{w}$, but $\vec{v} \neq \vec{w}$.

D2 First, I notice that $|\ ||\vec{x}|| - ||\vec{y}||\ | = ||\vec{x}|| - ||\vec{y}||$ if $||\vec{x}|| \geq ||\vec{y}||$, and $|\ ||\vec{x}|| - ||\vec{y}||\ | = ||\vec{y}|| - ||\vec{x}||$ if $||\vec{x}|| \leq ||\vec{y}||$. So, let's consider the first case: $||\vec{x}|| \geq ||\vec{y}||$. Then $|\ ||\vec{x}|| - ||\vec{y}||\ | = ||\vec{x}|| - ||\vec{y}|| = ||\vec{x} - \vec{y} + \vec{y}|| = ||\vec{y}|| \leq ||\vec{x} - \vec{y}|| + ||\vec{y}|| - ||\vec{y}||$ (by the triangle inequality) $= ||\vec{x} - \vec{y}||$, as desired. So that only leaves us with the case that $||\vec{x}|| \leq ||\vec{y}||$, which gives us $|\ ||\vec{x}|| - ||\vec{y}||\ | = ||\vec{y}|| - ||\vec{x}|| = ||\vec{y} - \vec{x} + \vec{x}|| - ||\vec{x}|| \leq ||\vec{y} - \vec{x}|| + ||\vec{x}|| - ||\vec{x}||$ (by the triangle inequality) $= ||\vec{y} - \vec{x}|| = ||(-1)(\vec{x} - \vec{y})|| = |-1|\ ||\vec{x} - \vec{y}||$ (by property 2 of theorem 2) $= ||\vec{x} - \vec{y}||$, as desired.

D3 To prove this fact, we'll make use of the definition that $||x|| = \sqrt{\vec{x} \cdot \vec{x}}$, or more specifically of the fact that $||x||^2 = \vec{x} \cdot \vec{x}$. So instead of trying to show that $||\vec{v}_1 + \vec{v}_2||^2 = ||\vec{v}_1||^2 + ||\vec{v}_2||^2$, I will show that $(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2) = \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2$.

$$\begin{array}{rcll}
(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2) & = & (\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_1 + (\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_2 & \text{Theorem 1, line (3)} \\
& = & \vec{v}_1 \cdot (\vec{v}_1 + \vec{v}_2) + \vec{v}_2 \cdot (\vec{v}_1 + \vec{v}_2) & \text{Theorem 1, line(2)} \\
& = & \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 & \text{Theorem 1, line (3)} \\
& = & \vec{v}_1 \cdot \vec{v}_1 + 0 + 0 + \vec{v}_2 \cdot \vec{v}_2 & \text{because } \vec{v}_1 \text{ and } \vec{v}_2 \text{ are orthogonal} \\
& = & \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 &
\end{array}$$