Solution to Practice 1j

A5(a) We first notice that the two given vectors are linearly independent. Since we are looking at two independent vectors, the span must be the equation of a plane. And as the given vectors are independent, they form a basis.

A5(b) Here we are given three independent vectors, so the given set is a basis. As there are three vectors, the span must be the equation of a hyperplane.

A5(c) Well, we immediately see that we can remove the vector $\vec{0}$, and another look shows that $\begin{bmatrix} 6 \\ 2 \\ -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. So, we have

$$\text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -2 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

So, $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for the set, and since we have only one vector, the set defines a line in $\mathbb{R}^4$.

D4(a) For a hyperplane, we will need three linearly independent vectors. The standard basis is always a nice place to start, so lets use $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as our three independent vectors. So all that's left to pick is a point, but we want to pick our point so that $\vec{0}$ IS NOT a solution to our equation. So, picking $\vec{p} = \vec{0}$ would obviously be a bad choice. But moreover, we need to pick $\vec{p}$ so that it is not a linear combination of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$. The choice that leaps to my mind at this point is our remaining basis vector–$\vec{e}_4$. So, an equation of a hyperplane not passing through the origin is $\vec{e}_4 + t_1\vec{e}_1 + t_2\vec{e}_2 + t_3\vec{e}_3$. This means that our solutions is

$$\vec{p} = \vec{e}_4 \quad \vec{v}_1 = \vec{e}_1 \quad \vec{v}_2 = \vec{e}_2 \quad \vec{v}_3 = \vec{e}_3$$

D4(b) For a plane passing through the origin we need two linearly independent vectors. Well, I don't see a reason not to use our standard basis vectors again! And since this time we DO want to pass through the origin, we can let $\vec{p} = \vec{0}$.

So, an equation for a plane passing through the origin is $\vec{x} = \vec{0} + t_1\vec{e}_1 + t_2\vec{e}_2$. This means that our solution is

$$\vec{p} = \vec{0} \quad \vec{v}_1 = \vec{e}_1 \quad \vec{v}_2 = \vec{e}_2 \quad \vec{v}_3 = \vec{0}$$

D4(c)

$$\vec{p} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_1 = \vec{0} \quad \vec{v}_2 = \vec{0} \quad \vec{v}_3 = \vec{0}$$

D4(d) A line only uses on vector, and we set $\vec{p} = \vec{0}$ so that we pass through the origin, which gives us the solution

$$\vec{p} = \vec{0} \quad \vec{v}_1 = \vec{e}_1 \quad \vec{v}_2 = \vec{0} \quad \vec{v}_3 = \vec{0}$$

D8(a) True, because $t\vec{v}_1 - \vec{v}_2 = \vec{0}$ is a non-trivial solution to the equation $t_1\vec{v}_1 + t_2\vec{v}_2 = \vec{0}$.

D8(b) True, because the only way to have $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent is to have there be an equation $t_1\vec{v}_1 + t_2\vec{v}_2 = \vec{0}$ where at least one of t_1 or t_2 is not zero. If we assume that $t_1 \neq 0$, then we get $t_1\vec{v}_1 = -t_2\vec{v}_2 \Rightarrow \vec{v}_1 = (-t_2/t_1)\vec{v}_2$, i.e. that $\vec{v}_1$ is a scalar multiple of $\vec{v}_2$.

D8(c) False. Consider the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$.

D8(d) True. Let $\vec{v}_1 = a\vec{v}_2 + bv_2$. Then $t_1 = 1$, $t_2 = -a$, and $t_3 = -b$ is a non-trivial solution to the equation $t_1\vec{v}_1 + t_2\vec{v}_2 + t_3\vec{v}_3 = \vec{0}$.

D8(e) False. If $\vec{v}_1 = \vec{0}$, then $\{\vec{v}_1\}$ is a subspace of $\mathbb{R}^4$.

D8(f) True, by theorem 2.

(**Course author's note:** My answer to D8(f) brings up a common question from students: "Do I need to memorize all the theorems?" Well yes, you're going to need to know them all, but to know them word for word, much less by the name "Theorem 2 from section 1.2" is not something I will expect of you. My rule of thumb is that if it has a name, you should be able to refer to it by name. (So, you should be able to say "because addition is commutative", for example.) But beyond that, you can simply refer to "a theorem". So, on an exam, the correct answer for D8(f) would be "True, by a theorem".)