Solution to Practice 1h

A2(e) Call the set E. Notice that $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, so $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} +$ $t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (t_1+1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (t_2+1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus, $E = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, and so by theorem 2, E is a subspace of $\mathbb{R}^3$.

A2(f) By theorem 2, $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^4$.

A6 Let $\vec{p}, \vec{d} \in \mathbb{R}^n$, and let $S = \{\vec{x} : \vec{x} = \vec{p} + t\vec{d} \text{ for some } t \in \mathbb{R}\}$.

($\Rightarrow$: Show that if S is a subspace, then $\vec{p}$ is a scalar multiple of $\vec{d}$.) Suppose that S is a subspace of $\mathbb{R}^n$. Then S is closed under scalar multiplication. Thus, given any $\vec{x} \in S$ we get that $2\vec{x} \in S$. But since $2\vec{x} = 2(\vec{p}+t\vec{d}) = s\vec{p}+st\vec{d} = \vec{p}+(\vec{p}+2t\vec{d})$, we must have that $\vec{p}+2t\vec{d} = r\vec{d}$ for some $r \in \mathbb{R}$. But this means that $\vec{p} = (r-2t)\vec{d}$, and so we see that $\vec{p}$ is a scalar multiple of $\vec{d}$.

(**Instructor's note**: In the back of the book, they use the fact that $\vec{0} \in S$ to show that $\vec{p}$ is a scalar multiple of $\vec{d}$, which I must admit is a simpler way to go. But I decided to go ahead and provide this solution, just to provide another perspective on the problem.)

($\Leftarrow$: Show that if $\vec{p}$ is a scalar multiple of $\vec{d}$, then S is a subspace of $\mathbb{R}^n$) Suppose $\vec{p}$ is a scalar multiple of $\vec{d}$, say $\vec{p} = r\vec{d}$. Then $\vec{p} + t\vec{d} = r\vec{d} + t\vec{d} = (r+t)\vec{d}$. Thus $S = \text{Span}\{\vec{d}\}$, and is therefore a subspace of $\mathbb{R}^n$.

D7 To prove that two sets are equal, we need to show that every element of the first set is an element of the second set, and vice versa. To that end, let $\vec{v_1}, \vec{v_2} \in \mathbb{R}^n$, and let s and t be FIXED real numbers, with $t \neq 0$.

($\Rightarrow$: Show that if $\vec{x} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}$, then $\vec{x} \in \text{Span}\{\vec{v_1}, s\vec{v_1} + t\vec{v_2}\}$) Suppose that $\vec{x} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}$. Then there are scalars a and b in $\mathbb{R}$ such that $\vec{x} = a\vec{v_1} + b\vec{v_2}$. Let $c = b/t$. Then $\vec{x} = a\vec{v_1} + ct\vec{v_2} = (a - cs)\vec{v_1} + c(s\vec{v_1} + t\vec{v_2})$, showing that $\vec{x} \in \text{Span}\{\vec{v_1}, s\vec{v_1} + t\vec{v_2}\}$. (**Instructor's note**: For the record, it took some scribbling on a piece of paper to come up with this value of c!)

($\Leftarrow$: Show that if $\vec{x} \in \text{Span}\{\vec{v_1}, s\vec{v_1} + t\vec{v_2}\}$, then $\vec{x} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}$) Suppose $\vec{x} \in \text{Span}\{\vec{v_1}, s\vec{v_1} + t\vec{v_2}\}$. Then there are scalars a and b in $\mathbb{R}$ such that $\vec{x} = a\vec{v_1} + b(s\vec{v_1} + t\vec{v_2})$. Then $\vec{x} = (a + bs)\vec{v_1} + bt\vec{v_2}$, so $\vec{x} \in \text{Span}\{\vec{v_1}, \vec{v_2}\}$.