Solution to Practice 1g

A2(a) Call the set A. Then A is not a subspace of $\mathbb{R}^3$ because $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in A$ (as $2^2 - 1^2 = 4 - 1 = 3$), but $2\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} \notin A$ (as $4^2 - 2^2 = 16 - 4 = 12 \neq 6$).

A2(b) Call the set B. Then $\vec{0} \in B$, so B in non-empty. Let $\vec{x}, \vec{y} \in B$. So $x_1 = x_3$ and $y_1 = y_3$. Then if we let $\vec{w} = \vec{x} + \vec{y}$ we get that $w_1 = x_1 + y_1 = x_3 + y_3 = w_3$. So $\vec{w} \in B$, and thus B is closed under addition. Finally, let $\vec{x} \in B$ (so $x_1 = x_3$), $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$. Then $z_1 = tx_1 = tx_3 = z_3$. So $\vec{z} \in B$, and thus B is closed under scalar multiplication. And as B is non-empty, closed under addition, and closed under scalar multiplication, we have that B is a subspace of $\mathbb{R}^3$.

A2(c) Call the set C. Then $\vec{0} \in C$, so C in non-empty. Next, let $\vec{x}, \vec{y} \in C$. So $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$. Then if we let $\vec{w} = \vec{x} + \vec{y}$, we get that $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$. Thus, $w_1 + w_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0 = 0$. So we have that $\vec{w} \in C$, which means that C is closed under scalar multiplication. Finally, let $\vec{x} \in C$ (so $x_1 + x_2 = 0$), $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$. Then $z_1 = tx_1$ and $z_2 = tx_2$, so $z_1 + z_2 = tx_1 + tx_2 = t(x_1 + x_2) = t(0) = 0$. So we have that $\vec{z} \in C$, which means that C is closed under scalar multiplication. And as C is non-empty, closed under addition, and closed under scalar multiplication, we have that C is a subspace of $\mathbb{R}^2$.

A2(d) Call the set D Then $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in D$, but $2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin D$, since $(2)(2) = 4 \neq 2$. So D is not closed under scalar multiplication, and thus D is not a subspace of $\mathbb{R}^3$.

A2(e) Call the set E. Then if we let $t_1 = t_2 = 0$ we see that $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \in E$, so E is non-empty. (**Instructor's note:** While we usually use $\vec{0}$ to show that E is non-empty, we can of course use any element of E to prove that E is non-empty.) Next, let $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, so that $\vec{x}, \vec{y} \in E$. Then if $\vec{w} = \vec{x} + \vec{y}$ we have:

$$\begin{aligned}
\vec{w} &= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (t_1 + s_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (t_2 + s_2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (t_1 + s_1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (t_2 + s_2) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (t_1 + s_1 + 1) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (t_2 + s_2 + 1) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}$$

And so, at long last, we see that $\vec{w} \in E$, so E is closed under addition. Lastly, let $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $s \in \mathbb{R}$ and let $\vec{z} = s\vec{x}$. Then

$$\begin{aligned}
\vec{z} &= s \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + st_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + st_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (s-1) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + st_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + st_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (s-1) \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) + st_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + st_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + (st_1 + (s-1)) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (st_2 + (s-1)) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}$$

This shows that $\vec{z} \in E$, and thus that E is closed under scalar multiplication. And as we have shown that E is non-empty, closed under addition, and closed under scalar multiplication, we have that E is a subspace of $\mathbb{R}^3$.

A3(a) Call the set A. Then $\vec{0} \in A$, so A is non-empty. Next, let $\vec{x}, \vec{y} \in A$, $t \in \mathbb{R}$, and let $\vec{w} = \vec{x} + \vec{y}$ and $\vec{z} = t\vec{x}$. Now, as $\vec{x}, \vec{y} \in A$, we have that $x_1 + x_2 + x_3 + x_4 = 0$ and $y_1 + y_2 + y_3 + y_4 = 0$. Then $w_1 + w_2 + w_3 + w_4 = (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + (x_4 + y_4) = (x_1 + x_2 + x_3 + x_4) + (y_1 + y_2 + y_3 + y_4) = 0 + 0 = 0$, so $\vec{w} \in A$, and thus A is closed under addition. And

$z_1+z_2+z_3+z_4 = tx_1+tx_2+tx_3+tx_4 = t(x_1+x_2+x_3+x_4) = t(0) = 0$, so $\vec{z} \in A$ and thus A is closed under scalar multiplication. And as we have shown that A is non-empty, closed under addition, and closed under scalar multiplication, we have that A is a subspace of $\mathbb{R}^4$.

A3(b) Call the set B. Then $\vec{0} \notin B$, so B is not a subspace of $\mathbb{R}^4$.

A3(c) Call the set C. Then $\vec{0} \notin C$, as $0 + 2(0) = 0 \neq 5$, so C is not a subspace of $\mathbb{R}^4$.

A3(d) Call the set D. Then $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in D$, as 1=(1)(1) and 1-1=0, but $2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \notin D$, as $2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ and $2 \neq (2)(2)$. As D is not closed under scalar multiplication, D is not a subspace of $\mathbb{R}^4$.

A3(e) Call the set E. Then $\vec{0} \in E$. Next, let $\vec{x}, \vec{y} \in E$, $t \in \mathbb{R}$, and let $\vec{w} = \vec{x} + \vec{y}$ and $\vec{z} = t\vec{x}$. Then $2x_1 = 3x_4$, $x_2 - 5x_3 = 0$, $2y_1 = 3y_4$, and $y_2 - 5y_3 = 0$. We also have that $2w_1 = 2(x_1 + y_1) = 2x_1 + 2y_1 = 3x_3 + 3y_4 = 3(x_4 + y_4) = 3w_4$, and $w_2 - 5w_3 = (x_2 + y_2) - 5(x_3 + y_3) = (x_2 - 5x_3) + (y_2 - 5y_3) = 0 + 0 = 0$, so $\vec{w} \in E$, and thus E is closed under addition. Similarly we see that $2z_1 = 2(tx_1) = t(2x_1) = t(3x_4) = 3(tx_4) = 3z_4$, and $z_2 - 5z_3 = tx_2 - 5(tx_3) = t(x_2 - 5x_3) = t(0) = 0$, so $\vec{z} \in E$, and thus E is closed under scalar multiplication. And as we have shown that E is non-empty, closed under addition, and closed under scalar multiplication, we have that E is a subspace of $\mathbb{R}^4$.

A3(f) Call the set F. Then $\vec{0} \notin F$, so F is not a subspace of $\mathbb{R}^4$.

D3(a) Let A be the intersection of U and V. That is, we have that $\vec{x} \in A$ if and only if $\vec{x} \in U$ AND $\vec{x} \in V$. Since U and V are both vector spaces, we have that $\vec{0} \in U$ and $\vec{0} \in V$, so $\vec{0} \in A$, and thus A is non-empty. Next, suppose that $\vec{x}$ and $\vec{y}$ are in A, and that $t \in \mathbb{R}$. As $\vec{x}$ and $\vec{y}$ are in A, they are also in U, which being a vector space is closed under addition and scalar multiplication. Thus, we have that both $\vec{x} + \vec{y} \in U$ and $t\vec{x} \in U$. Similarly, we have that $\vec{x}$ and $\vec{y}$ are in the vector space V, so $\vec{x} + \vec{y} \in V$ and $t\vec{x} \in V$. Thus we see that $\vec{x} + \vec{y} \in A$ and $t\vec{x} \in A$, giving us that A is closed under addition and scalar multiplication. And therefore A is a subspace of $\mathbb{R}^n$.

D3(b) Let $B = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and let $C = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the union of B and C (as it is in B), and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the union of B and C (as it is in C), but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the union of B and C, as $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin B$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin C$. Thus, the union of B and C is not closed under addition, and therefore is not a subspace.

D3(c) First we see that $U + V$ is non-empty, since $\vec{0} \in U$ and $\vec{0} \in V$ gives us $\vec{0} = \vec{0} + \vec{0} \in U + V$. So, let $\vec{x_1}, \vec{x_2} \in U + V$ and $t \in \mathbb{R}$, and let $\vec{u_1}, \vec{u_2} \in U$ and $\vec{v_1}, \vec{v_2} \in V$ be such that $\vec{x_1} = \vec{u_1} + \vec{v_1}$ and $\vec{x_2} = \vec{u_2} + \vec{v_2}$. Then $\vec{x_1} + \vec{x_2} = \vec{u_1} + \vec{v_1} + \vec{u_2} + \vec{v_2} = (\vec{u_1} + \vec{u_2}) + (\vec{v_1} + \vec{v_2})$. And since $\vec{u_1} + \vec{u_2} \in U$ and $\vec{v_1} + \vec{v_2} \in V$ we see that $\vec{x_1} + \vec{x_2} \in U + V$, and thus that $U + V$ is closed under addition. Finally, we have that $t\vec{x_1} = t(\vec{u_1} + \vec{v_1}) = t\vec{u_1} + t\vec{v_1}$. Since $t\vec{u_1} \in U$ and $t\vec{v_1} \in V$, we see that $t\vec{x_1} \in U + V$, and thus that $U + V$ is closed under scalar multiplication. And since $U + V$ is non-empty, closed under addition, and closed under scalar multiplication, we have that $U + V$ is a subspace of $\mathbb{R}^n$.