Lecture 1m

The Scalar Equation of a Hyperplane

(pages 34-36)

In keeping with the trend for this chapter, we will develop the general scalar equation of a hyperplane by first focusing our attention on the equation of a hyperplane in $\mathbb{R}^3$. As I mentioned in Lecture 1j, a hyperplane in $\mathbb{R}^3$ is the same as a plane, which is why this part of the textbook starts by discussing the scalar equation of a plane. But the results of this part of the textbook ONLY APPLY TO $\mathbb{R}^3$–the more general result ends up with the equation of a hyperplane in $\mathbb{R}^n$, not the equation of a plane in $\mathbb{R}^n$.

With that in mind, let's focus our attention on $\mathbb{R}^3$ for the moment. Previously we saw the vector equation of a plane as $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2$, where $\vec{p}$, $\vec{v}_1$, and $\vec{v}_2$ are all fixed vectors, and as we vary the scalars t_1 and t_2 we get all the vectors $\vec{x}$ on the plane. But it happens that we can also get a scalar equation for a plane, of the form $ax_1 + bx_2 + cx_3 = d$, where $a, b, c, d \in \mathbb{R}$, and the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is on the plane if and only if x_1, x_2, and x_3 satisfy the equation. Now, the scalar equation can be harder to compute, but it has the advantage that it is much easier to determine if a point is on the plane using a scalar equation (just plug in) verses using the vector equation (hunt to see if you can find the right values of t_1 and t_2, or prove that none exist). And when we move on to general $\mathbb{R}^n$, you would need to sort through $n-1$ t values! So, while the vector equation is usually easy to generate, and it is a great way to produce points on a plane/hyperplane, the scalar equation is more useful if you are trying to check whether or not a specific point is on your plane/hyperplane.

Well, now that I've got you interested, I should actually go about computing the scalar equations! The key to why we can find scalar equations of hyperplanes, and not surfaces in general, is that they are exactly one dimension smaller than the space they live in. As such, instead of listing the $n-1$ directions that ARE in the space, we instead look at the one direction that IS NOT in the hyperplane. And so, focusing our attention back on $\mathbb{R}^3$ for now, instead of describing a plane using the two direction vectors $\vec{v}_1$ and $\vec{v}_2$ that are in the plane, we are going to look for a third vector $\vec{n}$ that is not in the plane. But, as you might imagine, we can't use just any vector that's not in the plane. The vector we are looking for is called the **normal vector** for the plane, and it has the property that it is orthogonal to any directed line segment in the plane. So, if $\vec{x}$ is on the plane $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2$, then the directed line segment $\vec{PX} = \vec{x} - \vec{p}$ is in the plane, and so we get $\vec{n} \cdot (\vec{p} - \vec{x}) = 0$. This means that $n_1(x_1 - p_1) + n_2(x_2 - p_2) + n_3(x_3 - p_3) = 0$, or that $n_1x_1 + n_2x_2 + n_3x_3 = n_1p_1 + n_2p_2 + n_3p_3$. So this means that, in my general scalar equation of a plane ($ax_1 + bx_2 + cx_3 = d$), we have $a = n_1$, $b = n_2$, $c = n_3$, and $d = \vec{n} \cdot \vec{p}$.

Example: Find the scalar equation of the plane in $\mathbb{R}^3$ that passes through the point $P(-3, 2, 7)$ and has normal vector $\vec{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

First I compute $\vec{n} \cdot \vec{p} = (3)(-3) + (2)(2) + (1)(7) = 5$. Then the equation is $3x_1 + 2x_2 + x_3 = 5$.

Two planes in $\mathbb{R}^3$ are defined to be **parallel** if the normal vector to one plane is a non-zero scalar multiple of the normal vector of the other plane. Given that a normal vector is easily read off of the scalar equation of a plane, this makes it easy to see just from the equations whether or not two planes are parallel.

Example: The plane $3x_1+2x_2+x_3 = 5$ is parallel to the plane $9x_1+6x_2+3x_3 = -5$, (because $3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$), but is not parallel to $3x_1 + 2x_2 - x_3 = 7$ (since $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \neq t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ for any $t \in \mathbb{R}$).

Of course, this definition of parallel coincides with the usual notion of parallel: two planes are parallel if they are the same, or if they have no points in common.

Example: The planes $A = 3x_1 + 2x_2 + x_3 = 5$ and $B = 9x_1 + 6x_2 + 3x_3 = -5$ are parallel. They are not the same plane, as $(1, 1, 0)$ is on plane A (since 3(1)+2(1)+1(0)=5), but it is not on plane B (since $9(1)+6(1)+3(0) = 15 \neq -5$). So, let (a, b, c) be a generic point on A. Then we have $3a + 2b + c = 5$, which means that $9a+6b+3c = 3(3a)+3(2b)+3(c) = 3(3a+2b+c) = 3(5) = 15 \neq -5$, so (a, b, c) is not on B. Similarly, if we look at a generic point (a', b', c') on B, then it satisfies $9a' + 6b' + 3c' = -5$. But this means that $3a' + 2b' + c' = (1/3)(9a' + 6b' + 3c') = (1/3)(-5) = -5/3 \neq 5$, so (a', b', c') is not on A. As such, we see that A and B do not have any points in common.

We also say that two planes are **orthogonal** to each other if their normal vectors are orthogonal to each other.

Example: The planes $3x_1 + 2x_2 + x_3 = 5$ and $-2x_1 + 4x_2 - 2x_3 = -3$ are orthogonal, because $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (3)(-2) + (2)(4) + (1)(-2) = 0$.

As that's pretty much everything for $\mathbb{R}^3$, we can now move on to $\mathbb{R}^n$. So, let A be a hyperplane in $\mathbb{R}^n$ that contains the point $\vec{p}$, and let $\vec{m}$ be a vector that is orthogonal to every directed line segment in A. Then the scalar equation for A is

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = \vec{m} \cdot \vec{p}$$

Example: Find the scalar equation of the hyperplane in $\mathbb{R}^5$ that has normal vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and passes through the point $P(-1, 1, 4, 2, -4)$.

Well, first we compute $\vec{m} \cdot \vec{p} = (1)(-1) + (2)(1) + (3)(4) + (4)(2) + (5)(-4) = 1$. Then the equation for the hyperplane is $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 1$.