Lecture 1l

Length and Dot Product in $\mathbb{R}^n$

(pages 31-34)

Okay, now that we eased into the notion of length and the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can expand our definitions to a general $\mathbb{R}^n$.

Definition Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Then the **dot product** of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Note: The dot product is also known as the **scalar product** or the **standard inner product**.

And now that we have a general definition of the dot product, we get the following "useful properties" theorem:

Theorem 1.3.1 Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

(1)$\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
(2)$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
(3)$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$
(4)$(t\vec{x}) \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$

The proof of these properties is pretty straightforward. I shall prove property (2) as an example:

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = y_1x_1 + y_2x_2 + \cdots + y_nx_n = \vec{y} \cdot \vec{x}$.

We define the dot product first, as we can use the dot product to get a quite nice definition of the length of a vector...

Definition Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then we define the **norm** or **length** of $\vec{x}$ by

$$||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Example: Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$. Then $||\vec{x}|| = \sqrt{1^2+2^2+3^2+4^2+5^2} = \sqrt{1+4+9+16+25} = \sqrt{55}$.

Example: The length from $P(2,-5,3,8)$ to $Q(7,6,-2,-3)$ is the length of the vector $\vec{PQ} = \begin{bmatrix} 7 \\ 6 \\ -2 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -5 \\ -11 \end{bmatrix}$, which is $\sqrt{5^2+11^2+(-5)^2+(-11)^2} = \sqrt{25+121+25+121} = \sqrt{292} = 2\sqrt{73}$.

And now that we have a general definition for the norm, we have a theorem listing some of its useful properties:

Theorem 1.3.2 Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

(1)$||\vec{x}|| \geq 0$ and $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}$
(2)$||t\vec{x}|| = |t|\ ||\vec{x}||$
(3)$|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||\ ||\vec{y}||$, with equality if and only if $\{\vec{x}, \vec{y}\}$ is linearly independent
(4)$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$

Property (3) is known as the **Cauchy-Schwarz Inequality**, and property (4) is known as the **Triangle Inequality**. I want to bring some extra notice to property (4), because it is quite natural to want to say $||\vec{x}+\vec{y}|| = ||\vec{x}||+||\vec{y}||$, but we see in property (4) that this would be INCORRECT. Consider the following example:

Example: Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\vec{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Then $\vec{x}+\vec{y} = \begin{bmatrix} 1-1 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$. So we have $||\vec{x}|| = \sqrt{1^2+2^2} = \sqrt{5}$, $||\vec{y}|| = \sqrt{(-1)^2+(3)^2} = \sqrt{10}$, and $||\vec{x}+\vec{y}|| = \sqrt{0^2+5^2} = 5 \neq \sqrt{5}+\sqrt{10}$.

Before we move on to the next major topic, we have a couple more definitions to throw at you:

Definition A vector $\vec{x} \in \mathbb{R}^n$ such that $||\vec{x}|| = 1$ is called a **unit vector**.

The value of a unit vector comes from the occasional desire to divide or multiply by $||\vec{x}||$, and 1 is just such a lovely number to multiply or divide by. But starting from an arbitrary vector $\vec{x}$, we can find a unit vector with the same direction as $\vec{x}$, given by

$$\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$$

Example: Find a unit vector in the direction of $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$.

First we need to compute $||\vec{x}|| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14}$. Then a unit vector in the direction of $\vec{x}$ is $\frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$.

Definition Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are **orthogonal** to each other if and only if $\vec{x} \cdot \vec{y} = 0$.

Note that this definition implies that $\vec{0}$ is orthogonal to every vector in $\mathbb{R}^n$. But if $\vec{x}$ and $\vec{y}$ are both not $\vec{0}$, then the notion of orthogonal is the same as the vectors $\vec{x}$ and $\vec{y}$ being perpendicular. That is, the angle between $\vec{x}$ and $\vec{y}$ is $\frac{\pi}{2}$, which we see using the formula for the angle between vectors.

Example: The vectors $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ are not orthogonal, because $\vec{x} \cdot \vec{y} = (1)(3) + (2)(-1) + (1)(-2) = -1$, but the vectors $\vec{w} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$ are orthogonal, because $\vec{w} \cdot \vec{z} = (4)(-3) + (2)(5) + (-2)(-1) = 0$.