Lecture 1j

Surfaces in Higher Dimensions

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Life in three dimensions is awfully nice. Want a line? Just find two points. Want a plane? Three non-collinear points or two lines will define one for you. But what about in higher dimensions. What is a "line" in ten dimensions? Ten dimensional space doesn't even exist! Well, mathematicians have never felt the need to be constrained by the real world, and so we figure that if we can define a line in $\mathbb{R}^2$ and $\mathbb{R}^3$, we ought to be able to define a line in any $\mathbb{R}^n$. And in that spirit, we have the following definitions:

<u>Definition</u> Let $\vec{p}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. Then we call the set with vector equation $\vec{x} = \vec{p} + t\vec{v},\ t \in \mathbb{R}$ a **line** in $\mathbb{R}^n$ that passes through $\vec{p}$, with direction vector $\vec{v}$.

<u>Definition</u> Let $\vec{v}_1, \vec{v}_2, \vec{p} \in \mathbb{R}^n$, with $\{\vec{v}_1, \vec{v}_2\}$ being a linearly independent set. Then the set with vector equation $\vec{x} = \vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2,\ t_1, t_2 \in \mathbb{R}$ is called a **plane** in $\mathbb{R}^n$ that passes through $\vec{p}$.

<u>Definition</u> Let $\vec{v}_1, \ldots, \vec{v}_{n-1}, \vec{p} \in \mathbb{R}^n$, with $\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$ being a linearly independent set. Then the set with vector equation $\vec{x} = \vec{p} + t_1\vec{v}_1 + \cdots + t_{n-1}\vec{v}_{n-1}$, $t_1, \ldots, t_{n-1} \in \mathbb{R}$ is called a **hyperplane** in $\mathbb{R}^n$ that passes through $\vec{p}$.

So, the general definition of a line goes with the "point and a direction" notion of a line. A plane is a point with two different directions, and a hyperplane is a point with one fewer direction than the dimension of the space. I'll point out that a line is the same as a hyperplane in $\mathbb{R}^2$ (there is no such thing as a plane in $\mathbb{R}^2$), and a plane is the same thing as a hyperplane in $\mathbb{R}^3$.

Example: Show that the set Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ defines a plane in $\mathbb{R}^4$.

First we notice that the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not linearly independent, since $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. So we remove the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and we still have $\text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix}\right\}$. Now $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix}\right\}$ is linearly independent, and we can rewrite $\text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix}\right\}$ as $\vec{x} = t_1\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + t_2\begin{bmatrix}0\\2\\0\\2\end{bmatrix}$. This means that $\text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix}\right\}$ is a plane in $\mathbb{R}^4$ going through $\vec{0}$ (where $\vec{0}$ was the missing $\vec{p}$ in the defining equation of a plane).

Example: Show that the set $\text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ defines a hyperplane in $\mathbb{R}^4$.

By “inspection” (again, we'll learn a way to formalize this later), we see that the set $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$ is linearly independent. So, we rewrite

$$\text{Span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\}$$

as the following equation of a hyperplane:

$$\vec{x} = \vec{0} + t_1\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + t_2\begin{bmatrix}0\\2\\0\\2\end{bmatrix} + t_3\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\ t_1, t_2, t_3 \in \mathbb{R}$$