Lecture 1i

Linear Independence and Basis

(pages p.20-23)

The idea that a spanning set may contain unnecessary vectors leads to the following definition.

Definition A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be **linearly dependent** if there exists coefficients $t_1, \ldots, t_k$ not all zero such that

$$\vec{0} = t_1\vec{v}_1 + \ldots + t_k\vec{v}_k$$

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be **linearly independent** if the only solution to

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is $t_1 = \cdots = t_k = 0$. This is called the **trivial solution**.

What we want to be clear about is that every set of vectors is either linearly dependent or linearly independent (and never both!). So when you start with a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$, we look for solutions to the equation $\vec{0} = t_1\vec{v}_1 + \ldots + t_k\vec{v}_k$. The first thing we notice is that $t_1 = \cdots = t_k = 0$ is ALWAYS a solution to this equation. That's why its called the "trivial" solution–it almost isn't worth mentioning. Now, if that is the ONLY solution to the equation, we fall into the "linearly independent" category. But if we can find even one other solution, that is a solution where at least one of the t_i is not zero, then we fall into the "linearly dependent" category.

Example: Show that the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

To do this, we need to find t_1, t_2, t_3 not all zero such that

$$t_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Later in the course we will look at a systematic way to solve this equation, but for now we will use "inspection" to notice that $t_1 = 5$, $t_2 = -2$, and $t_3 = 1$ is a non-trivial solution to this equation.

Example: Show that the set $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is linearly independent.

To do this, we need to show that the only solution to the equation

$$t_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is $t_1 = t_2 = 0$. To that end, lets break this vector equation into its components, creating the following system of equations:

$$2t_1 + t_2 = 0 \qquad\qquad t_1 + 2t_2 = 0$$

From the second equation, we see that $t_1 = -2t_2$, which when we plug into the first equation gives us $2(-2t_2) + t_2 = 0 \Rightarrow -3t_2 = 0 \Rightarrow t_2 = 0$. And since $t_1 = -2t_2$, we now have $t_1 = -2(0) = 0$. Thus, we see that the only solution to the equation $t_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $t_1 = t_2 = 0$, which means that the set $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is linearly independent.

So why do we care? Well, let's look at what happens with the span of a linearly dependent set. Let's say that we have coefficients $t_1, \ldots, t_k$ not all zero such that $\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$, and lets further assume that $t_k \neq 0$. (We know that at least on $t_i \neq 0$, so we can simply rearrange the order of our vectors to make $t_k \neq 0$.) Then $t_k\vec{v}_k = -t_1\vec{v}_1 - t_2\vec{v}_2 - \cdots - t_{k-1}\vec{v}_{k-1}$, so $\vec{v}_k = -(t_1/t_k)\vec{v}_1 - (t_2/t_k)\vec{v}_2 - \cdots - (t_{k-1}/t_k)\vec{v}_{k-1}$ (see, we're making use of the fact that $t_k \neq 0$), which means that $\vec{v}_k$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_{k-1}$. So, by Theorem 1.2.3 we have $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$.

Now, at this point we might still have that $\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ is linearly dependent, but by repeatedly removing dependent members, we will eventually end up with a linearly independent set. The advantage of a linearly independent set is that it is somehow more simple than a dependent set. It is not cluttered with unnecessary information. And mathematicians do not like their lives to be cluttered with unnecessary information! But before we move on to the pinnacle of sets (yes, it gets better than simply being independent), I'll give a passing mention to Theorem 4.

Theorem 1.2.4: If a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ contains the zero vector, then it is linearly dependent.

Personally, I'm not sure that this rates "Theorem" status, but it is certainly a useful fact to keep in mind. But now, lets move on to the next topic. (The previously mentioned pinnacle of sets...)

Definition: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a spanning set for a subspace S of $\mathbb{R}^n$ and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is called a **basis** for S.

The idea behind a basis is that it contains just the right balance of vectors–there are enough vectors to make sure that we span S (the condition that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a spanning set for a subspace S), but it doesn't contain any unnecessary vectors to accomplish this goal (the condition that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent). But I wish to note that a basis for S is not a unique entity. For one thing, you could replace any vector $\vec{v}_i$ with any of its non-zero scalar multiples $s\vec{v}_i$ and you would still have a basis. I also want to note that the notion of a basis will be extremely, but not until later in the course. For now, however, we have one last example to look at, that is so important it comes in the form of a definition.

Definition: In $\mathbb{R}^n$, let $\vec{e}_i$ represent the vector whose i-th component is 1 and all other components are 0. The set $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is called the **standard basis for** $\mathbb{R}^n$.