Lecture 1h

Spanning Sets

(pages 18-20)

A common way to define a subspace is through using a spanning set. But before we get to this definition, we note the following.

Theorem 1.2.2 If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$ and S is the set of all possible linear combinations of these vectors, that is

$$S = \{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k | t_1, \dots, t_k \in \mathbb{R}\}$$

then S is a subspace of $\mathbb{R}^n$

The proof of this theorem is straightforward and available in the text. But I want us to focus on the uses of this theorem. To that end, we have the following definition.

Definition If S is the subspace of $\mathbb{R}^n$ consisting of all possible linear combinations of the vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then S is called the subspace **spanned** by the set of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$, and we say that the set $\mathcal{B}$ **spans** S. The set $\mathcal{B}$ is called a **spanning set** for the subspace S. We write

$$S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\mathcal{B}$$

Example: Some elements of Span$\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ are

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad -2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad \tfrac{1}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad 0\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice that Span$\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ = Span $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ = Span $\left\{t\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ for any $t \neq 0$.

Some elements not in Span$\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ are

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We see from this that Span$\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\} \neq \mathbb{R}^2$

Example: We have that Span$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^3$, since for any

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ we have that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example: Notice that $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.
To demonstrate this, we need to show that every element of the set on the left is also an element of the set on the right, and vice-versa:

$$\vec{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow \quad \text{there are } t_1, t_2, t_3 \in \mathbb{R} \text{ such that } \vec{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\Rightarrow \quad \vec{x} = (t_1 + t_3) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (t_2 + t_3) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and

$$\vec{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow \quad \text{there are } t_1, t_2 \in \mathbb{R} \text{ such that } \vec{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

This example is part of a more general phenomenon, where if one member of the spanning set is a linear combination of the other elements, then it is essentially unnecessary and can be removed. This process is formalized by the following theorem:

Theorem 1.2.3 Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. If $\vec{v}_k$ can be written as a linear combination of $\vec{v}_1, \ldots \vec{v}_{k-1}$, then

$$\text{Span}\{\vec{v}_1 \ldots \vec{v}_k\} = \text{Span}\{\vec{v}_1 \ldots \vec{v}_{k-1}\}$$

As the order is which we list the vectors doesn't matter, it isn't necessarily "$\vec{v}_k$" that we remove. By rearranging we could remove any $\vec{v}_i$ that can be written as a linear combination of the others.

Example: $\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \right\}$, because $\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$.