Lecture 1g

Subspaces

(pages 16-17)

I will begin our discussion of subspaces with the following definition of a subspace, which has been modified from the one given in the text.

Definition A subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if the following conditions hold:

(0) S is non-empty

(1) S is closed under addition (that is, for $\vec{x}, \vec{y} \in S$ we have $\vec{x} + \vec{y} \in S$)

(2) S is closed under scalar multiplication
(that is, for $t \in \mathbb{R}$ and $\vec{x} \in S$, we have $t\vec{x} \in S$)

This definition is of course the same as the one given in the text, but I prefer to present the definition in this manner as it emphasizes the often overlooked requirement that S be non-empty. In fact, a subspace must always contain $\vec{0}$ (which follows from property (2), using $t = 0$), so frequently the first thing one looks for when trying to determine if a subset S is in fact a subspace is whether or not $\vec{0} \in S$. Moreover, one can quickly prove that the single element subset $\{\vec{0}\}$ is always a subspace of $\mathbb{R}^n$, and that $\mathbb{R}^n$ is always a subspace of $\mathbb{R}^n$ as well. For other subsets, we need to pay closer attention to properties (1) and (2). At first glance, you may not understand why these properties don't always hold. After all, of course the sum of two vectors is a vector. The key is the fact that we have S written in the places where we have previously written the all inclusive $\mathbb{R}^n$. So, when we take two elements of S and add them, we don't want to end up with some random vector from $\mathbb{R}^n$. We want to make sure that our result is still contained in S. And this needs to be true for all possible combinations of vectors in S! And then even if that works, we need to make sure that all the scalar multiples are contained in S as well.

With the exception of the set $\{\vec{0}\}$, subspaces will always contain an infinite number of elements. As such, we can't describe S with a list of elements. So, our definitions of S will always be abstract. I find that many students stumble at this point, and the only advice I have is to look at lots of examples (coming next!).

Example 1 Consider the subset $S_1 = \{\vec{0}, \vec{e_1}, \vec{e_2}\}$ of $\mathbb{R}^2$. Then S_1 is non-empty (and even contains $\vec{0}$). But S_1 is not closed under addition, as $\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$. S_1 is also not closed under scalar multiplication, as $2\vec{e_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, but $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin S$. So S_1 is NOT a subspace of $\mathbb{R}^2$.

Before I move on to more examples, there are two things that I want to point out about how my example differs from a "proof" that S_1 is not a subspace. The first is that I looked at all three properties, when all that is needed to show that S_1 is not a subspace is that any one of the properties fails. So, for example, the last two sentences would have been enough for full marks. The second thing that I want to emphasize is that to show that S_1 did not have properties (1) and (2), I provided specific counterexamples. In theory a definition for S could be so abstract that you would need to give a completely abstract argument for why one of these properties does not hold, but for this course you should always be able to give me either two specific elements of S whose sum is not in S or a specific element of S and a specific element of $\mathbb{R}$ whose scalar product is not in S. So, with that advice in hand, lets look at some more subsets.

Example 2 Consider the subset $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1 = 1 \right\}$ of $\mathbb{R}^2$. Then $\vec{0} \notin S_2$, so S_2 is not a subspace of $\mathbb{R}^2$. (While technically $\vec{0} \notin S$ is not one of the properties we need, we previously discussed why every subspace of $\mathbb{R}^n$ must contain $\vec{0}$, so this would be an acceptable answer.)

Example 3 Consider the subset $S_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} | x_1 x_2 - x_3 = 0 \right\}$. Then $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \in S_3$ as $(2)(3) - 6 = 0$, but $2 \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} \notin S_3$, as $(4)(6) - 12 = 12 \neq 0$. So S_3 is not a subspace of $\mathbb{R}^3$.

Example 4 Consider the subset $S_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1, x_2 \leq 0 \right\}$ of $\mathbb{R}^2$. Then $\begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S_4$, but $(-1) \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S_4$. So S_4 is not a subspace of $\mathbb{R}^2$.

Okay, now lets look at some things that ARE subsets.

Example 5 Consider the subset $S_5 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | x_1 = 0 \right\}$ of $\mathbb{R}^2$. Then $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S_5$, so S_5 is non-empty. So let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be elements of S_5. Then $x_1 = y_1 = 0$. Let $\vec{z} = \vec{x} + \vec{y}$. Then $\vec{z} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 + 0 \\ x_2 + y_2 \end{bmatrix} =$

$\begin{bmatrix} 0 \\ x_2 + y_2 \end{bmatrix}$. Since $z_1 = 0$ we see that $\vec{z} \in S_5$, and thus S_5 is closed under addition. Finally, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S_5$ and $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$. Then $\vec{z} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = \begin{bmatrix} t(0) \\ tx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ tx_2 \end{bmatrix}$. Since $z_1 = 0$ we have that $\vec{z} \in S_5$, and thus S_5 is closed under scalar multiplication. And as S_5 is non-empty, closed under addition, and closed under scalar multiplication, we have that S_5 is a subspace of $\mathbb{R}^2$.

Example 6 Consider the subset $S_6 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} | x_3 = x_1 + x_2 \right\}$ of $\mathbb{R}^3$. Then $\vec{0} \in S_6$ since 0=0+0. Next, suppose that $\vec{x}, \vec{y} \in S_6$. Then we have $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$. So let $\vec{z} = \vec{x} + \vec{y}$. Then $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, and $z_3 = x_3 + y_3 = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) = z_1 + z_2$. So we have $\vec{z} \in S_6$, and thus that S_6 is closed under addition. Finally, let $\vec{x} \in S_6$ (so $x_3 = x_1 + x_2$), $t \in \mathbb{R}$, and let $\vec{z} = t\vec{x}$. Then $z_1 = tx_1$, $z_2 = tx_2$, and $z_3 = tx_3 = t(x_1 + x_2) = tx_1 + tx_2 = z_1 + z_2$. So we see that $\vec{z} \in S_6$, and thus that S_6 is closed under scalar multiplication. And as we have shown that S_6 is non-empty, closed under addition, and closed under scalar multiplication, we have that S_6 is a subspace of $\mathbb{R}^3$.