Lecture 1f

Vectors in $\mathbb{R}^n$

(pages 14-16)

So far we've been easing you into the world of vectors, but at long last it is time to proceed to the general world of $\mathbb{R}^n$, for a n a positive integer.

Definition $\mathbb{R}^n$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $x_i \in \mathbb{R}$. In set notation, we write

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} | x_1, \ldots, x_n \in \mathbb{R} \right\}$$

As in $\mathbb{R}^2$ and $\mathbb{R}^3$, we define addition and scalar multiplication of vectors in $\mathbb{R}^n$ componentwise:

Definition If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\in \mathbb{R}^n$ and $t \in \mathbb{R}$, then we define addition of vectors by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

and we define scalar multiplication by

$$t\vec{x} = t = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix}$$

From these definitions, we get the following theorem:

Theorem 1.2.1 For all vectors $\vec{w}$, $\vec{x}$, and $\vec{y}$ in $\mathbb{R}^n$, and all scalars s and t in $\mathbb{R}$, we have the following 10 properties:

Property 1: $\vec{x} + \vec{y} \in \mathbb{R}^n$, or in words, we say $\mathbb{R}^n$ is closed under addition.

This property is obviously true, based on our definition of vector addition, but the point was not so much whether this property was in doubt, but rather an emphasis that our definition of addition was a sensible one.

Property 2: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$, or in words, we say addition is commutative.

Property 3: $(\vec{x}+\vec{y})+\vec{w} = \vec{x}+(\vec{y}+\vec{w})$, or in words, we say addition is associative.

Combined, properties 2 and 3 emphasize that we can add vectors in any order that we want.

Property 4: There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^n$, or in words, we say that $\mathbb{R}^n$ has a zero vector.

This property emphasizes the fact that we can add the zero vector without changing anything.

Property 5: For each $\vec{x} \in \mathbb{R}^n$, there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x}+(-\vec{x}) = \vec{0}$. In words, we say that $\mathbb{R}^n$ has additive inverses.

Once we have property 5, we gain the ability to solve equations. For example, lets try to find $\vec{z}$ such that

$$\vec{z} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

We can solve this by adding the inverse of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to both sides:

$$\left(\vec{z} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

Using our associative property on the left, this becomes

$$\vec{z} + \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) = \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

Our additive inverse property makes this

$$\vec{z} + \vec{0} = \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \left(-\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

And our zero vector property makes this

$$\vec{z} = \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \left(- \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Now, our additive inverse property merely tells us that $-\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ exists, but we of course know that $-\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and so we have that

$$\vec{z} = \begin{bmatrix} -6 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

I should note at this point that you will not be expected to list every property you use when solving a vector equation. Instead, the point of this theorem and our current discussion of it, is to emphasize our ability to perform the sort of calculations we are used to performing from our study of the real numbers.

But, our list of properties is not yet complete, and so we move on to:

Property 6: $t\vec{x} \in \mathbb{R}^n$, or in words, we say $\mathbb{R}^n$ is closed under scalar multiplication.

This property is similar to Property 1, in that is simply points out that we have a reasonable definition for scalar multiplication.

Property 7: $s(t\vec{x}) = (st)\vec{x}$, or in words, we say scalar multiplication is associative.

Since we also know that $st = ts$ from our knowledge of the real numbers, this property tells us that we can multiply scalars by a vector in any order.

The next two properties are known as the distributive laws:

Property 8: $(s + t)\vec{x} = s\vec{x} + t\vec{x}$, or in words, we say we can distribute a vector.

Property 9: $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$, or in words, we say we can distribute a scalar.

And finally, we have our last property:

Property 10: $1\vec{x} = \vec{x}$, or in words, there is a scalar multiplicative identity.

That said, for all its importance, the proof of the theorem is really easy, but tedious. However, it does give a nice introduction to the way we prove statements about vectors. As an example, and I will give a proof of Property 9.

Proof of Theorem 1.2.1 (9) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}
t(\vec{x}+\vec{y}) &= t\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) \\
&= t\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \\
&= \begin{bmatrix} t(x_1 + y_1) \\ \vdots \\ t(x_n + y_n) \end{bmatrix} \\
&= \begin{bmatrix} tx_1 + ty_1 \\ \vdots \\ tx_n + ty_n \end{bmatrix} \\
&= \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} + \begin{bmatrix} ty_1 \\ \vdots \\ ty_n \end{bmatrix} \\
&= t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + t\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= t\vec{x} + t\vec{y}
\end{aligned}$$

Now, not only does this demonstrate the common practice of proving something about vectors by looking at the individual components, it also demonstrates the CORRECT way to do a "prove the left side is equal to the right side" proof. However, it often happens that there is a critical step in such a proof where a leap of creativity is needed. So frequently we do such proofs in two columns, which can also work, if done as follows:

Another proof of Theorem 1.2.1, (9)

$$t(\vec{x}+\vec{y}) \qquad\qquad t\vec{x}+t\vec{y}$$

$$= t\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) \qquad\qquad = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + t\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} t(x_1+y_1) \\ \vdots \\ t(x_n+y_n) \end{bmatrix} \qquad\qquad = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} + \begin{bmatrix} ty_1 \\ \vdots \\ ty_n \end{bmatrix}$$

$$= \begin{bmatrix} tx_1+ty_1 \\ \vdots \\ tx_n+ty_n \end{bmatrix} \qquad\qquad = \begin{bmatrix} tx_1+ty_1 \\ \vdots \\ tx_n+ty_n \end{bmatrix}$$

Thus we have $t(\vec{x}+\vec{y}) = \begin{bmatrix} tx_1+ty_1 \\ \vdots \\ tx_n+ty_n \end{bmatrix} = t\vec{x}+t\vec{y}$, as desired.

The advantage of this method is that you can work from both sides of the equations, looking for a common point that can link them together. But I find that students often want to connect these two columns with an "=" all the way down. As the point of the proof is to prove that the left side equals the right side, it is incorrect to place such an equal sign until you have actually demonstrated it is valid to do so. Which is why I prefer the single column technique, but as you've seen the two column technique works just as well so long as you keep the two columns completely separate, and then link them together in the rest of your proof.

To finish our introduction to $\mathbb{R}^n$, we extend our definition of $\vec{0}$ to $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, and we will write $\vec{e_i}$ for the vector with a "1" in the i^{th} component and a "0" in the other components. As before, we will let context determine which specific version of $\vec{0}$ or $\vec{e_i}$ we are referring to. We also continue to write $-\vec{x}$ for $(-1)\vec{x}$ and $\vec{x}-\vec{y}$ for $\vec{x}+(-1)\vec{y}$.