Solution to Practice 5e

B1(a) Since $\begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4-2 \\ -5-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we see that the first column of P is not an eigenvector of A, and thus P does not diagonalize A.

B1(b) Since $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. Since $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 3-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we see that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is also an eigenvector. As such, we know that P diagonalizes A. We find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \begin{array}{c} \\ (-1/2)R_2 \end{array}$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right].$$

So $P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$. From our previous calculations, we know that $AP = \begin{bmatrix} 4 & -2 \\ 4 & 2 \end{bmatrix}$. As such, we have:

$$\begin{aligned} P^{-1}AP &= P^{-1}(AP) \\ &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+2 & -1+1 \\ 2-2 & -1-1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

And so we see that $P^{-1}AP$ is a diagonal matrix.

B1(c) Since $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+6 \\ 6+6 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we see that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector. Since $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 3-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we see that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is also an eigenvector. As such, we know that P diagonalizes A. We find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} (1/2)R_1 \\ \end{array} \sim \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & -5/2 & -3/2 & 1 \end{array}\right] \; (-2/5)R_2 \quad \sim \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 3/5 & -2/5 \end{array}\right] \; R_1 - (1/2)R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 1/5 \\ 0 & 1 & 3/5 & -2/5 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix}$. From our previous calculations, we know that $AP = \begin{bmatrix} 8 & -1 \\ 12 & 1 \end{bmatrix}$. As such, we have:

$$\begin{aligned} P^{-1}AP &= P^{-1}(AP) \\ &= \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 12 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (8+12)/5 & (-1+1)/5 \\ (24-24)/5 & (-3-2)/5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

And so we see that $P^{-1}AP$ is a diagonal matrix.

B1(d) Since $\begin{bmatrix} -7 & 2 & -4 \\ 8 & -1 & 4 \\ 18 & -6 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7+2-12 \\ -8-1+12 \\ -18-6+33 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$, we see that $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ is an eigenvector. Since $\begin{bmatrix} -7 & 2 & -4 \\ 8 & -1 & 4 \\ 18 & -6 & 11 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7+4-12 \\ -8-2+12 \\ -18-12+33 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, we see that $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ is also an eigenvector. Since $\begin{bmatrix} -7 & 2 & -4 \\ 8 & -1 & 4 \\ 18 & -6 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7-2+8 \\ 8+1-8 \\ 18+6-22 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, we see that $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ is also an eigenvector. As such, we know that P diagonalizes A. We find P^{-1} using the matrix inverse algorithm:

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \; -R_1 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \; \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right] \; R_1 + R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right] \; R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

So $P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. From our previous calculations, we know that $AP = \begin{bmatrix} -3 & -1 & -1 \\ 3 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix}$. As such, we have:

$$\begin{aligned} P^{-1}AP &= P^{-1}(AP) \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -1 & -1 \\ 3 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -3-3+9 & -1-2+3 & -1-1+2 \\ -3+3+0 & -1+2+0 & -1+1+0 \\ -9+0+9 & -3+0+3 & -3+0+2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

And so we see that $P^{-1}AP$ is a diagonal matrix.

B2(a) First we need to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} -4-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (-4-\lambda)(1-\lambda) - 6 \\ &= -4 + 4\lambda - \lambda + \lambda^2 - 6 \\ &= -10 + 3\lambda + \lambda^2 \\ &= (2-\lambda)(-5-\lambda) \end{aligned}$$

So the eigenvalues of A are $\lambda = 2$ and $\lambda = -5$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 2$, we need to find the general solution to $(A - 2I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 2I$ as follows:

$$\begin{bmatrix} -4-2 & 3 \\ 2 & 1-2 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \; R_1 \updownarrow R_2 \; \sim \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \; R_2 + 3R_1$$
$$\sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $2v_1 - v_2 = 0$, or $v_2 = 2v_1$. If we replace the variable v_1 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 2$ is Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. To find the eigenspace for $\lambda = -5$, we need to find the general solution to $(A + 5I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + 5I$ as follows:

$$\begin{bmatrix} -4+5 & 3 \\ 2 & 1+5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \; R_2 - 2R_1 \; \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + 3v_2 = 0$, or $v_1 = -3v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3s \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -5$ is Span$\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$.

Since the eigenvalues of A have equal algebraic and geometric multiplicity, it is diagonalizable, with $P = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$.

B2(b) First we need to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} 5-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} \\ &= (5-\lambda)(3-\lambda) \end{aligned}$$

So the eigenvalues of A are $\lambda = 5$ and $\lambda = 3$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 5I$ as follows:

$$\begin{bmatrix} 5-5 & 2 \\ 0 & 3-5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \; 1/2R_1 \sim \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \; R_2 + 2R_1 \; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_2 = 0$. If we replace the variable v_1 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 5$ is Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. To find the eigenspace for $\lambda = 3$, we need to find the general solution to $(A - 3I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 3I$ as follows:

$$\begin{bmatrix} 5-3 & 2 \\ 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + v_2 = 0$, or $v_1 = -v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 3$ is Span$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Since the eigenvalues of A have equal algebraic and geometric multiplicity, it is diagonalizable, with $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

B2(c) First we need to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 5 & 2 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 5-\lambda & 2 \\ 0 & 5-\lambda \end{bmatrix} \\ &= (5-\lambda)(5-\lambda) \end{aligned}$$

So the only eigenvalue of A is $\lambda = 5$, which has algebraic multiplicity 2. To find the eigenspace for $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 5I$ as follows:

$$\begin{bmatrix} 5-5 & 2 \\ 0 & 5-5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{matrix} 1/2R_1 \\ \end{matrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_2 = 0$. If we replace the variable v_1 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 5$ is Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, and so we see that the geometric multiplicity of $\lambda = 5$ is only 1. Since this is not the same as its algebraic multiplicity, A is not diagonalizable.

B3(a) First we need to find the eigenvalues of A:

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\
&= \det \begin{bmatrix} 4-\lambda & 3 \\ 5 & 6-\lambda \end{bmatrix} \\
&= (4-\lambda)(6-\lambda) - 15 \\
&= 24 - 4\lambda - 6\lambda + \lambda^2 - 15 \\
&= 9 - 10\lambda + \lambda^2 \\
&= (1-\lambda)(9-\lambda)
\end{aligned}$$

So the eigenvalues of A are $\lambda = 1$ and $\lambda = 9$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 1$, we need to find the general solution to $(A - I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - I$ as follows:

$$\begin{bmatrix} 4-1 & 3 \\ 5 & 6-1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \begin{matrix} (1/3)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 5R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + v_2 = 0$, or $v_1 = -v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 1$ is Span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. To find the eigenspace for $\lambda = 9$, we need to find the general solution to $(A - 9I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 9I$ as follows:

$$\begin{bmatrix} 4-9 & 3 \\ 5 & 6-9 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 5 & -3 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \sim \begin{bmatrix} -5 & 3 \\ 0 & 0 \end{bmatrix} \begin{matrix} (-1/5)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 - (3/5)v_2 = 0$, or $v_1 = (3/5)v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (3/5)s \\ s \end{bmatrix} = s \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 9$ is Span $\left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$.

Since the eigenvalues of A have equal algebraic and geometric multiplicity, it is diagonalizable, with $P = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$.

B3(b) First we need to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 3 & -6 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 3-\lambda & -6 \\ -4 & 8-\lambda \end{bmatrix} \\ &= (3-\lambda)(8-\lambda) - 24 \\ &= 24 - 3\lambda - 8\lambda + \lambda^2 - 24 \\ &= -11\lambda + \lambda^2 \\ &= -\lambda(11-\lambda) \end{aligned}$$

So the eigenvalues of A are $\lambda = 0$ and $\lambda = 11$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 0$, we need to find the general solution to $(A - (0)I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - (0)I = A$ as follows:

$$\begin{bmatrix} 3 & -6 \\ -4 & 8 \end{bmatrix} \begin{matrix} (1/3)R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \begin{matrix} \\ R_2 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 - 2v_2 = 0$, or $v_1 = 2v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 0$ is $\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. To find the eigenspace for $\lambda = 11$, we need to find the general solution to $(A - 11I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 11I$ as follows:

$$\begin{bmatrix} 3-11 & -6 \\ -4 & 8-11 \end{bmatrix} = \begin{bmatrix} -8 & -6 \\ -4 & -3 \end{bmatrix} \begin{matrix} (-1/8)R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & 3/4 \\ -4 & -3 \end{bmatrix} \begin{matrix} \\ R_2 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 3/4 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + (3/4)v_2 = 0$, or $v_1 = (-3/4)v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (-3/4)s \\ s \end{bmatrix} = s\begin{bmatrix} -3/4 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 11$ is Span $\left\{ \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$.

Since the eigenvalues of A have equal algebraic and geometric multiplicity, it is diagonalizable, with $P = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \\ 0 & 11 \end{bmatrix}$.

B3(c) First we need to find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} 1-\lambda & -4 \\ -4 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)(1-\lambda) - 16 \\ &= 1 - \lambda - \lambda + \lambda^2 - 16 \\ &= -15 - 2\lambda + \lambda^2 \\ &= (-3-\lambda)(5-\lambda) \end{aligned}$$

So the eigenvalues of A are $\lambda = -3$ and $\lambda = 5$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = -3$, we need to find the general solution to $(A + 3I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + 3I$ as follows:

$$\begin{bmatrix} 1+3 & -4 \\ -4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \begin{matrix} (1/4)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{matrix} \\ R_2 + 4R_1 \end{matrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 - v_2 = 0$, or $v_1 = v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -3$ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. To find the eigenspace for $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 5I$ as follows:

$$\begin{bmatrix} 1-5 & -4 \\ -4 & 1-5 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} \begin{matrix} (-1/4)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + v_2 = 0$, or $v_1 = -v_2$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 5$ is Span $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Since the eigenvalues of A have equal algebraic and geometric multiplicity, it is diagonalizable, with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$.

B3(d) First we need to find the eigenvalues of A:

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} -7 & 2 & 12 \\ -3 & 0 & 6 \\ -3 & 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\
&= \det \begin{bmatrix} -7-\lambda & 2 & 12 \\ -3 & -\lambda & 6 \\ -3 & 1 & 5-\lambda \end{bmatrix} \\
& \quad \text{(replace row 2 with row 2 plus row 3)} \\
&= \det \begin{bmatrix} -7-\lambda & 2 & 12 \\ 0 & -1-\lambda & 1+\lambda \\ -3 & 1 & 5-\lambda \end{bmatrix} \\
& \quad \text{(expanding along the second row)} \\
&= 0 + (-1-\lambda) \begin{vmatrix} -7-\lambda & 12 \\ -3 & 5-\lambda \end{vmatrix} - (1+\lambda) \begin{vmatrix} -7-\lambda & 2 \\ -3 & 1 \end{vmatrix} \\
&= (-1-\lambda)((-7-\lambda)(5-\lambda) + 36) - (1+\lambda)((-7-\lambda) + 6) \\
&= (-1-\lambda)(-35 + 7\lambda - 5\lambda + \lambda^2 + 36) + (-1-\lambda)(-1-\lambda) \\
&= (-1-\lambda)(1 + 2\lambda + \lambda^2 - 1 - \lambda) \\
&= (-1-\lambda)(\lambda + \lambda^2) \\
&= -\lambda(-1-\lambda)^2
\end{aligned}$$

So the eigenvalues of A are $\lambda = 0$, which has algebraic multiplicity 1, and $\lambda = -1$, which has algebraic multiplicity 2. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 0$, we need to find the general solution to $(A + (0)I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + (0)I = A$ as follows:

$$\begin{bmatrix} -7 & 2 & 12 \\ -3 & 0 & 6 \\ -3 & 1 & 5 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} -3 & 0 & 6 \\ -7 & 2 & 12 \\ -3 & 1 & 5 \end{bmatrix} \begin{matrix} (-1/3)R_1 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ -7 & 2 & 12 \\ -3 & 1 & 5 \end{bmatrix} \begin{matrix} \\ R_2 + 7R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix} (1/2)R_2 \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equations $v_1 - 2v_3 = 0$ (or $v_1 = 2v_3$), and $v_2 - v_3 = 0$ (or $v_2 = v_3$). If we replace the variable v_3 with the parameter s, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 0$ is Span $\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\}$, and we see that $\lambda = 0$ has geometric multiplicity equal to its algebraic multiplicity. To find the eigenspace for $\lambda = -1$, we need to find the general solution to $(A+I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + I$ as follows:

$$\begin{bmatrix} -7+1 & 2 & 12 \\ -3 & 1 & 6 \\ -3 & 1 & 5+1 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 12 \\ -3 & 1 & 6 \\ -3 & 1 & 6 \end{bmatrix} \begin{matrix} (-1/6)R_1 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & -1/3 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 6 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + 3R_1 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & -1/3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 - (1/3)v_2 - 2v_3 = 0$, or $v_1 = (1/3)v_2 + 2v_3$. If we replace the variable v_2 with the parameter s and the variable v_3 with the parameter t, then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (1/3)s + 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = s\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the eigenspace for $\lambda = -1$ is Span $\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$, and we see that $\lambda = -1$ has geometric multiplicity equal to its algebraic multiplicity. Thus, A is diagonalizable, with $P = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

B3(e) First we need to find the eigenvalues of A:

$$
\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix} 3 & 0 & -4 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \det\begin{bmatrix} 3-\lambda & 0 & -4 \\ 1 & 1-\lambda & -2 \\ 1 & 0 & -1-\lambda \end{bmatrix} \\
& \quad \text{(expanding along the second column)} \\
&= (1-\lambda)\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} \\
&= (1-\lambda)((3-\lambda)(-1-\lambda)+4) \\
&= (1-\lambda)(-3-3\lambda+\lambda+\lambda^2+4) \\
&= (1-\lambda)(1-2\lambda+\lambda^2) \\
&= (1-\lambda)^3
\end{aligned}
$$

So the only eigenvalue of A is $\lambda = 1$, which has algebraic multiplicity 3. To find the eigenspace for $\lambda = 1$, we need to find the general solution to $(A-I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - I$ as follows:

$$
\begin{bmatrix} 3-1 & 0 & -4 \\ 1 & 1-1 & -2 \\ 1 & 0 & -1-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -4 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

So our system is equivalent to the equation $v_1 - 2v_3 = 0$, or $v_1 = 2v_3$. If we replace the variable v_2 with the parameter s and the variable v_3 with the parameter t, then we see that the general solution is

$$
\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}
$$

Thus, the eigenspace for $\lambda = 1$ is Span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$, and we see that $\lambda = 1$ has geometric multiplicity equal 2. Since this is not the same as its algebraic multiplicity, A is not diagonalizable.

B3(f) First we need to find the eigenvalues of A:

$$
\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix} \\
& \quad \text{(expanding along the third row)} \\
&= 0 + 0 + (-2-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \\
&= (-2-\lambda)((1-\lambda)(4-\lambda)-4) \\
&= (-2-\lambda)(4-\lambda-4\lambda+\lambda^2-4) \\
&= (-2-\lambda)(-5\lambda+\lambda^2) \\
&= -\lambda(-2-\lambda)(5-\lambda)
\end{aligned}
$$

So the eigenvalues of A are $\lambda = 0$, $\lambda = -2$, and $\lambda = 5$. Next, we need to find their eigenspaces. To find the eigenspace for $\lambda = 0$, we need to find the general solution to $(A + (0)I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + (0)I = A$ as follows:

$$
\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ (-1/2)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; R_2 \updownarrow R_3 \; \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

So our system is equivalent to the equations $v_1 + 2v_2 = 0$ (or $v_1 = -2v_2$), and $v_3 = 0$. If we replace the variable v_2 with the parameter s, then we see that the general solution is

$$
\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}
$$

Thus, the eigenspace for $\lambda = 0$ is Span $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$. To find the eigenspace for $\lambda = -2$, we need to find the general solution to $(A + 2I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A + 2I$ as follows:

$$
\begin{bmatrix} 1+2 & 2 & 0 \\ 2 & 4+2 & 0 \\ 0 & 0 & -2+2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; R_1 \updownarrow R_2 \; \sim \begin{bmatrix} 2 & 6 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; (1/2)R_1
$$

$$
\sim \begin{bmatrix} 1 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; R_3 - 3R_1 \; \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; (-1/7)R_2
$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equations $v_1 = 0$ and $v_2 = 0$. If we replace the variable v_3 with the parameter s then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -2$ is Span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. To find the eigenspace for $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$. To do this, we need to row reduce $A - 5I$ as follows:

$$\begin{bmatrix} 1-5 & 2 & 0 \\ 2 & 4-5 & 0 \\ 0 & 0 & -2-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{matrix} (-1/4)R_1 \\ \\ (-1/7)R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 - 2R_1 \sim \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equations $v_1 - (1/2)v_2 = 0$ (or $v_1 = (1/2)v_2$), and $v_3 = 0$. If we replace the variable v_2 with the parameter s then we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (1/2)s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 5$ is Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$. Since all the eigenvalues of A have geometric multiplicity equal to its algebraic multiplicity, A is diagonalizable, with $P = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.