Solution to Practice 5d

B3(a) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$\begin{aligned} C(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 3-\lambda & 0 \\ 0 & 7-\lambda \end{bmatrix} \\ &= (3-\lambda)(7-\lambda) \end{aligned}$$

And so we see that the eigenvalues are $\lambda = 3, 7$. Moreover, we see that the algebraic multiplicity of $\lambda = 3$ is 1, and the algebraic multiplicity of $\lambda = 7$ is 1. To find their geometric multiplicity, we need to find their eigenspaces.

To find the eigenspace for $\lambda = 3$, we need to find the general solution to $(A - 3I)\vec{v} = \vec{0}$, which we do by row reducing $A - 3I$:

$$\begin{bmatrix} 3-3 & 0 \\ 0 & 7-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} \begin{matrix} \\ (1/7)R_2 \end{matrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_2 = 0$. Replacing the variable v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 3$ is Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 3$ is 1.

To find the eigenspace for $\lambda = 7$, we need to find the general solution to $(A - 7I)\vec{v} = \vec{0}$, which we do by row reducing $A - 7I$:

$$\begin{bmatrix} 3-7 & 0 \\ 0 & 7-7 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} (-1/4)R_1 \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_1 = 0$. Replacing the variable v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 7$ is Span$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 7$ is 1.

B3(b) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$\begin{aligned} C(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} -3 & -3 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} -3-\lambda & -3 \\ 0 & -3-\lambda \end{bmatrix} \\ &= (-3-\lambda)^2 \end{aligned}$$

And so we see that the only eigenvalue is $\lambda = -3$, and that the algebraic multiplicity of $\lambda = -3$ is 2. To find their geometric multiplicity, we need to find the eigenspace. To do this, we need to find the general solution to $(A+3I)\vec{v} = \vec{0}$, which we do by row reducing $A + 3I$:

$$\begin{bmatrix} -3+3 & -3 \\ 0 & -3+3 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{matrix} (-1/3)R_1 \\ \end{matrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_2 = 0$. Replacing the variable v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = -3$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = -3$ is 1.

B3(c) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$\begin{aligned} C(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 7 & 3 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 7-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix} \\ &= (7-\lambda)(2-\lambda) - (2)(3) \\ &= 14 - 7\lambda - 2\lambda + \lambda^2 - 6 = 8 - 9\lambda + \lambda^2 \\ &= (1-\lambda)(8-\lambda) \end{aligned}$$

And so we see that the eigenvalues are $\lambda = 1, 8$. Moreover, we see that the algebraic multiplicity of $\lambda = 1$ is 1, and the algebraic multiplicity of $\lambda = 8$ is 1. To find their geometric multiplicity, we need to find their eigenspaces.

To find the eigenspace for $\lambda = 1$, we need to find the general solution to $(A - I)\vec{v} = \vec{0}$, which we do by row reducing $A - I$:

$$\begin{bmatrix} 7-1 & 3 \\ 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \begin{matrix} (1/6)R_1 \\ \ \end{matrix} \sim \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix} \begin{matrix} \ \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_1 + (1/2)v_2 = 0$, or $v_1 = (-1/2)v_2$. Replacing the variable v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (-1/2) \\ s \end{bmatrix} = s \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 1$ is $\text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 1$ is 1.

To find the eigenspace for $\lambda = 8$, we need to find the general solution to $(A - I)\vec{v} = \vec{0}$, which we do by row reducing $A - I$:

$$\begin{bmatrix} 7-8 & 3 \\ 2 & 2-8 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix} \begin{matrix} -R_1 \\ \ \end{matrix} \sim \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \begin{matrix} \ \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_1 - 3v_2 = 0$, or $v_1 = 3v_2$. Replacing the variable v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 8$ is $\text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 8$ is 1.

B3(d) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$\begin{aligned} C(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} -4 & 0 & 0 \\ 2 & -8 & 4 \\ -4 & 5 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\ &= \begin{bmatrix} -4-\lambda & 0 & 0 \\ 2 & -8-\lambda & 4 \\ -4 & 5 & -\lambda \end{bmatrix} \text{ (expanding along the first row)} \\ &= (-4-\lambda) \begin{vmatrix} -8-\lambda & 4 \\ 5 & -\lambda \end{vmatrix} \\ &= (-4-\lambda)((-8-\lambda)(-\lambda) - 20) = (-4-\lambda)(-20 + 8\lambda + \lambda^2) \\ &= (-4-\lambda)(2-\lambda)(-10-\lambda) \end{aligned}$$

And so we see that the eigenvalues are $\lambda = 2, -4, -10$. Moreover, we see that the algebraic multiplicity of $\lambda = 2$ is 1, the algebraic multiplicity of $\lambda = -4$ is 1, and the algebraic multiplicity of $\lambda = -10$ is 1. To find their geometric multiplicity, we need to find their eigenspaces.

To find the eigenspace for $\lambda = 2$, we need to find the general solution to $(A - 2I)\vec{v} = \vec{0}$, which we do by row reducing $A - 2I$:

$$\begin{bmatrix} -4-2 & 0 & 0 \\ 2 & -8-2 & 4 \\ -4 & 5 & -2 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 2 & -10 & 4 \\ -4 & 5 & -2 \end{bmatrix} \begin{matrix} (-1/6)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -10 & 4 \\ -4 & 5 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -10 & 4 \\ 0 & 5 & -2 \end{bmatrix} (-1/10)R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/5 \\ 0 & 5 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 5R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/5 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$v_1 = 0, \qquad v_2 - (2/5)v_3 = 0 \text{ (or } v_2 = (2/5)v_3)$$

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 2/5 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 2/5 \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 2$ is $\text{Span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 2$ is 1.

To find the eigenspace for $\lambda = -4$, we need to find the general solution to $(A + 4I)\vec{v} = \vec{0}$, which we do by row reducing $A + 4I$:

$$\begin{bmatrix} -4+4 & 0 & 0 \\ 2 & -8+4 & 4 \\ -4 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 4 \\ -4 & 5 & 4 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 2 & -4 & 4 \\ 0 & 0 & 0 \\ -4 & 5 & 4 \end{bmatrix} R_2 \updownarrow R_3$$

$$\sim \begin{bmatrix} 2 & -4 & 4 \\ -4 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ -4 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix} R_2 + 4R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & 0 \end{bmatrix} (-1/3)R_2 \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$v_1 - 6v_3 = 0 (\text{or } v_1 = 6v_3), \qquad v_2 - 4v_3 = 0 \text{ (or } v_2 = 4v_3)$$

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 6s \\ 4s \\ s \end{bmatrix} = s \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -4$ is Span $\left\{ \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = -4$ is 1.

To find the eigenspace for $\lambda = -10$, we need to find the general solution to $(A + 10I)\vec{v} = \vec{0}$, which we do by row reducing $A + 10I$:

$$\begin{bmatrix} -4+10 & 0 & 0 \\ 2 & -8+10 & 4 \\ -4 & 5 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 2 & 4 \\ -4 & 5 & 10 \end{bmatrix} \begin{matrix} (1/6)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ -4 & 5 & 10 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 5 & 10 \end{bmatrix} (1/2)R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 5 & 10 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 5R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$v_1 = 0, \qquad v_2 + 2v_3 = 0 \text{ (or } v_2 = -2v_3)$$

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -10$ is Span $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = -10$ is 1.

B3(e) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$C(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} \text{ (expanding along the first row)}$$

$$= (4-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 4-\lambda \\ 2 & 2 \end{vmatrix}$$
$$= (4-\lambda)((4-\lambda)(4-\lambda)-4) - 2(2(4-\lambda)-4) + 2(4-2(4-\lambda))$$
$$= (4-\lambda)(16-8\lambda+\lambda^2-4) - 2(8-2\lambda-4) + 2(4-8+2\lambda)$$
$$= 48 - 32\lambda + 4\lambda^2 - 12\lambda + 8\lambda^2 - \lambda^3 - 8 + 4\lambda - 8 + 4\lambda$$
$$= 32 - 36\lambda + 12\lambda^2 - \lambda^3$$
$$= (2-\lambda)^2(8-\lambda)$$

And so we see that the eigenvalues are $\lambda = 2, 8$. Moreover, we see that the algebraic multiplicity of $\lambda = 2$ is 2 and the algebraic multiplicity of $\lambda = 8$ is 1. To find their geometric multiplicity, we need to find their eigenspaces.

To find the eigenspace for $\lambda = 2$, we need to find the general solution to $(A - 2I)\vec{v} = \vec{0}$, which we do by row reducing $A - 2I$:

$$\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \\ \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the equation $v_1 + v_2 + v_3 = 0$, or $v_1 = -v_2 - v_3$. Replacing the variable v_2 with the parameter s and the variable v_3 with the parameter t, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 2$ is Span $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. As the eigenspace has only two basis vectors, the geometric multiplicity of $\lambda = 2$ is 2.

To find the eigenspace for $\lambda = 8$, we need to find the general solution to $(A - 8I)\vec{v} = \vec{0}$, which we do by row reducing $A - 8I$:

$$\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 2 & 2 & -4 \\ 2 & -4 & 2 \\ -4 & 2 & 2 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 2 & -4 & 2 \\ -4 & 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix} \begin{matrix} \\ (-1/6)R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 6 & -6 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - 6R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$v_1 - v_3 = 0 \text{ (or } v_1 = v_3), \qquad v_2 - v_3 = 0 \text{ (or } v_2 = v_3)$$

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = 8$ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 8$ is 1.

B3(f) The first thing we need to do is find the eigenvalues. To do that, we need to factor the characteristic polynomial:

$$
\begin{aligned}
C(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} -9 & -7 & 7 \\ -9 & -7 & 2 \\ 8 & -8 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \begin{bmatrix} -9-\lambda & -7 & 7 \\ -9 & -7-\lambda & 2 \\ 8 & -8 & -3-\lambda \end{bmatrix} \\
& \quad \text{(expanding along the first row)} \\
&= (-9-\lambda)\begin{vmatrix} -7-\lambda & 2 \\ -8 & -3-\lambda \end{vmatrix} + 7\begin{vmatrix} -9 & 2 \\ 8 & -3-\lambda \end{vmatrix} \\
& \quad +7\begin{vmatrix} -9 & -7-\lambda \\ 8 & -8 \end{vmatrix} \\
&= (-9-\lambda)((-7-\lambda)(-3-\lambda)+16) + 7(-9(-3-\lambda)-16) \\
& \quad +7(72-8(-7-\lambda)) \\
&= (-9-\lambda)(21+7\lambda+3\lambda+\lambda^2+16) + 7(27+9\lambda-16) \\
& \quad +7(72+56+8\lambda) \\
&= (-9-\lambda)(37+10\lambda+\lambda^2) + 7(11+9\lambda) + 7(128+8\lambda) \\
&= -333-90\lambda-9\lambda^2-37\lambda-10\lambda^2-\lambda^3+77+63\lambda+896+56\lambda \\
&= 640-8\lambda-19\lambda^2-\lambda^3 \\
&= (5-\lambda)(-8-\lambda)(-16-\lambda)
\end{aligned}
$$

And so we see that the eigenvalues are $\lambda = 5, -8, -16$. Moreover, we see that the algebraic multiplicity of $\lambda = 5$ is 1, the algebraic multiplicity of $\lambda = -8$ is 1, and the algebraic multiplicity of $\lambda = -16$ is 1. To find their geometric multiplicity, we need to find their eigenspaces.

To find the eigenspace for $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$, which we do by row reducing $A - 5I$:

$$
\begin{bmatrix} -9-5 & -7 & 7 \\ -9 & -7-5 & 2 \\ 8 & -8 & -3-5 \end{bmatrix} = \begin{bmatrix} -14 & -7 & 7 \\ -9 & -12 & 2 \\ 8 & -8 & -8 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_3 \\ \\ \\ \end{matrix}
$$

$$
\sim \begin{bmatrix} 8 & -8 & -8 \\ -9 & -12 & 2 \\ 14 & -7 & 7 \end{bmatrix} \begin{matrix} (1/8)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ -9 & -12 & 2 \\ 14 & -7 & 7 \end{bmatrix} \begin{matrix} \\ R_2 + 9R_1 \\ R_3 + 14R_1 \end{matrix}
$$

$$
\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -21 & -7 \\ 0 & -21 & -7 \end{bmatrix} (-1/21)R_2 \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/3 \\ 0 & -21 & -7 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + 21R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}
$$

And so we see that our system is equivalent to the system

$v_1 - (2/3)v_3 = 0$ (or $v_1 = (2/3)v_3$), $\qquad v_2 + (1/3)v_3 = 0$ (or $v_2 = (-1/3)v_3$)

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (2/3)s \\ (-1/3)s \\ s \end{bmatrix} = s\begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = 5$ is Span $\left\{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = 5$ is 1.

To find the eigenspace for $\lambda = -8$, we need to find the general solution to $(A + 8I)\vec{v} = \vec{0}$, which we do by row reducing $A + 8I$:

$$\begin{bmatrix} -9+8 & -7 & 7 \\ -9 & -7+8 & 2 \\ 8 & -8 & -3+8 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 7 \\ -9 & 1 & 2 \\ 8 & -8 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 9R_1 \\ R_3 + 8R_1 \end{matrix}$$
$$\sim \begin{bmatrix} -1 & -7 & 7 \\ 0 & 64 & -61 \\ 0 & -64 & 61 \end{bmatrix} (1/64)R_2 \sim \begin{bmatrix} -1 & -7 & 7 \\ 0 & 1 & -61/64 \\ 0 & -64 & 61 \end{bmatrix} \begin{matrix} R_1 + 7R_2 \\ \\ R_3 + 64R_3 \end{matrix} \sim$$
$$\begin{bmatrix} -1 & 0 & 21/64 \\ 0 & 1 & -61/64 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$-v_1 + (21/64)v_3 = 0 (\text{or } v_1 = (21/64)v_3), \qquad v_2 - (61/64)v_3 = 0 \ (\text{or } v_2 = (61/64)v_3)$$

Replacing the variable v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (21/64)s \\ (61/64)s \\ s \end{bmatrix} = s\begin{bmatrix} 21/64 \\ 61/64 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 21/64 \\ 61/64 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 21 \\ 61 \\ 64 \end{bmatrix}\right\}$$

Thus, the eigenspace for $\lambda = -8$ is Span $\left\{\begin{bmatrix} 21 \\ 61 \\ 64 \end{bmatrix}\right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = -8$ is 1.

To find the eigenspace for $\lambda = -16$, we need to find the general solution to $(A + 16I)\vec{v} = \vec{0}$, which we do by row reducing $A + 16I$:

$$\begin{bmatrix} -9+16 & -7 & 7 \\ -9 & -7+16 & 2 \\ 8 & -8 & -3+16 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 7 \\ -9 & 9 & 2 \\ 8 & -8 & 13 \end{bmatrix} (1/7)R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -9 & 9 & 2 \\ 8 & -8 & 13 \end{bmatrix} \begin{matrix} \\ R_2 + 9R_1 \\ R_3 - 8R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 11 \\ 0 & 0 & 5 \end{bmatrix} \; (1/11)R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - 5R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

And so we see that our system is equivalent to the system

$$v_1 - v_2 = 0 \text{ (or } v_1 = v_2), \qquad v_3 = 0$$

Replacing the variable v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus, the eigenspace for $\lambda = -16$ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. As the eigenspace has only one basis vector, the geometric multiplicity of $\lambda = -16$ is 1.