Solution to Practice 5c

B2(a) We already know that the eigenvalues are $\lambda = 3, 6$, and that $A - \lambda I = \begin{bmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = 3$ we need to find the general solution to $(A - 3I)\vec{v} = \vec{0}$, which means we need to row reduce $A - 3I = \begin{bmatrix} 4-3 & -1 \\ -2 & 5-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 - v_2 = 0$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

So, the eigenspace for $\lambda = 3$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

To find the eigenspace for $\lambda = 6$ we need to find the general solution to $(A - 6I)\vec{v} = \vec{0}$, which means we need to row reduce $A - 6I = \begin{bmatrix} 4-6 & -1 \\ -2 & 5-6 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $-2v_1 - v_2 = 0$, or $v_2 = -2v_1$. Replacing v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ -2s \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$$

So, the eigenspace for $\lambda = 6$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$.

B2(b) We already know that the only eigenvalue is $\lambda = 3$, and that $B - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = 3$ we need to find the general solution to $(B -$

$3I)\vec{v} = \vec{0}$, which means we need to row reduce $B - 3I = \begin{bmatrix} 2-3 & 1 \\ -1 & 4-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $-v_1 + v_2 = 0$, or $v_1 = v_2$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = 3$ is Span$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

B2(c) We already know that the eigenvalues are $\lambda = -1, 4$, and that $C - \lambda I = \begin{bmatrix} -2-\lambda & 2 \\ -3 & 5-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = -1$ we need to find the general solution to $(C+I)\vec{v} = \vec{0}$, which means we need to row reduce $C+I = \begin{bmatrix} -2+1 & 2 \\ -3 & 5+1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \end{matrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $-v_1 + 2v_2 = 0$, or $v_1 = 2v_2$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = -1$ is Span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

To find the eigenspace for $\lambda = 4$ we need to find the general solution to $(C - 4I)\vec{v} = \vec{0}$, which means we need to row reduce $C - 4I = \begin{bmatrix} -2-4 & 2 \\ -3 & 5-4 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ \end{matrix} \sim \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $3v_1 - v_2 = 0$, or $v_2 = 3v_1$. Replacing v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ 3s \end{bmatrix} = s\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

So, the eigenspace for $\lambda = 4$ is Span $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

B2(d) We already know that the eigenvalues are $\lambda = 1, -4$, and that $D - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ -3 & -5-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = 1$ we need to find the general solution to $(D - I)\vec{v} = \vec{0}$, which means we need to row reduce $D - I = \begin{bmatrix} 2-1 & 2 \\ -3 & -5-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $v_1 + 2v_2 = 0$, or $v_1 = -2v_2$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

So, the eigenspace for $\lambda = 1$ is Span $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

To find the eigenspace for $\lambda = -4$ we need to find the general solution to $(D + 4I)\vec{v} = \vec{0}$, which means we need to row reduce $D + 4I = \begin{bmatrix} 2+4 & 2 \\ -3 & -5+4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -3 & -1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} 6 & 2 \\ -3 & -1 \end{bmatrix} \begin{matrix} (1/2)R_2 \\ \end{matrix} \sim \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

So our system is equivalent to the equation $3v_1 + v_2 = 0$, or $v_2 = -3v_1$. Replacing v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ -3s \end{bmatrix} = s\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$$

So, the eigenspace for $\lambda = -4$ is Span $\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$.

B2(e) We already know that the eigenvalues are $\lambda = 1, 2, 3$, and that $E - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 5 \\ 0 & 2-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = 1$, we need to find the general solution to $(E - I)\vec{v} = \vec{0}$, which means that we need to row reduce $E - I = \begin{bmatrix} 1-1 & 3 & 5 \\ 0 & 2-1 & 7 \\ 0 & 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} 0 & 3 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 0 & 1 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & -16 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ (-1/16)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} R_1 - 7R_2 \\ \\ R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So our system is equivalent to the system

$$v_2 = 0 \qquad v_3 = 0$$

Replacing v_1 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = 1$ is $\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

To find the eigenspace for $\lambda = 2$, we need to find the general solution to $(E - 2I)\vec{v} = \vec{0}$, which means that we need to row reduce $E - 2I = \begin{bmatrix} 1-2 & 3 & 5 \\ 0 & 2-2 & 7 \\ 0 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ (1/7)R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 5R_2 \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the system

$$-v_1 + 3v_2 = 0 \text{ (or } v_1 = 3v_2) \qquad v_3 = 0$$

Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = 2$ is Span$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find the eigenspace for $\lambda = 3$, we need to find the general solution to $(E - 3I)\vec{v} = \vec{0}$, which means that we need to row reduce $E - 3I = \begin{bmatrix} 1-3 & 3 & 5 \\ 0 & 2-3 & 7 \\ 0 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -2 & 3 & 5 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 3R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} -2 & 0 & 26 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ -R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the system

$$v_1 - 13v_3 = 0 \text{ (or } v_1 = 13v_3) \qquad v_2 - 7v_3 = 0 \text{ (or } v_2 = 7v_3)$$

Replacing v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 13s \\ 7s \\ s \end{bmatrix} = s \begin{bmatrix} 13 \\ 7 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 13 \\ 7 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = 3$ is Span$\left\{ \begin{bmatrix} 13 \\ 7 \\ 1 \end{bmatrix} \right\}$.

B2(f) We already know that the eigenvalues are $\lambda = -1, -2, 2$, and that $F - \lambda I = \begin{bmatrix} -4-\lambda & 6 & 6 \\ -2 & 2-\lambda & 4 \\ -1 & 3 & 1-\lambda \end{bmatrix}$.

To find the eigenspace for $\lambda = -1$, we need to find the general solution to $(F + I)\vec{v} = \vec{0}$, which means that we need to row reduce $F+I = \begin{bmatrix} -4+1 & 6 & 6 \\ -2 & 2+1 & 4 \\ -1 & 3 & 1+1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -3 & 6 & 6 \\ -2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix} \begin{matrix} (-1/3)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & -2 \\ -2 & 3 & 4 \\ -1 & 3 & 2 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ -R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the system

$$v_1 - 2v_3 = 0 \text{ (or } v_1 = 2v_3) \qquad v_2 = 0$$

Replacing v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = -1$ is $\text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

To find the eigenspace for $\lambda = -2$, we need to find the general solution to $(F + 2I)\vec{v} = \vec{0}$, which means that we need to row reduce $F+2I = \begin{bmatrix} -4+2 & 6 & 6 \\ -2 & 2+2 & 4 \\ -1 & 3 & 1+2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 6 \\ -2 & 4 & 4 \\ -1 & 3 & 3 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -2 & 6 & 6 \\ -2 & 4 & 4 \\ -1 & 3 & 3 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 4 \\ -1 & 3 & 3 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} (-1/2)R_2 \sim \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 3R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the system

$$v_1 = 0 \qquad v_2 + v_3 = 0 \text{ (or } v_2 = -v_3)$$

Replacing v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = -2$ is Span $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

To find the eigenspace for $\lambda = 2$, we need to find the general solution to $(F - 2I)\vec{v} = \vec{0}$, which means that we need to row reduce $F - 2I = \begin{bmatrix} -4-2 & 6 & 6 \\ -2 & 2-2 & 4 \\ -1 & 3 & 1-2 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 6 \\ -2 & 0 & 4 \\ -1 & 3 & -1 \end{bmatrix}$. We row reduce as follows:

$$\begin{bmatrix} -6 & 6 & 6 \\ -2 & 0 & 4 \\ -1 & 3 & -1 \end{bmatrix} \begin{matrix} (-1/6)R_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ -2 & 0 & 4 \\ -1 & 3 & -1 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} (-1/2)R_2 \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to the system

$$v_1 - 2v_3 = 0 \text{ (or } v_1 = 2v_3) \qquad v_2 - v_3 = 0 \text{ (or } v_2 = v_3)$$

Replacing v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So, the eigenspace for $\lambda = 2$ is Span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.