Solution to Practice 5b

B2(a) To find the eigenvalues of $A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$, we need to compute $\det(A - \lambda I) = \det\left(\begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 4-\lambda & -1 \\ -2 & 5-\lambda \end{bmatrix} =$ $(4-\lambda)(5-\lambda)-(-2)(-1) = 20-4\lambda-5\lambda+\lambda^2-2 = 18-9\lambda+\lambda^2 = (3-\lambda)(6-\lambda)$. As such, $\det(A-\lambda I) = 0$ if and only if $\lambda = 3, 6$, and so the eigenvalues of A are $\lambda = 3, 6$.

B2(b) To find the eigenvalues of $B = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$, we need to compute $\det(B - \lambda I) = \det\left(\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} =$ $(2-\lambda)(4-\lambda)-(-1)(1) = 8-2\lambda-4\lambda+\lambda^2+1 = 9-6\lambda+\lambda^2 = (3-\lambda)^2$. As such, $\det(B-\lambda I) = 0$ if and only if $\lambda = 3$, and so the only eigenvalue of B is $\lambda = 3$.

B2(c) To find the eigenvalues of $C = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix}$, we need to compute $\det(C - \lambda I) = \det\left(\begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -2-\lambda & 2 \\ -3 & 5-\lambda \end{bmatrix} =$ $(-2-\lambda)(5-\lambda)-(-3)(2) = -10+2\lambda-5\lambda+\lambda^2+6 = -4-3\lambda+\lambda^2 = (4-\lambda)(-1-\lambda)$. As such, $\det(C-\lambda I) = 0$ if and only if $\lambda = -1, 4$, and so the eigenvalues of C are $\lambda = -1, 4$.

B2(d) To find the eigenvalues of $D = \begin{bmatrix} 2 & 2 \\ -3 & -5 \end{bmatrix}$, we need to compute $\det(D - \lambda I) = \det\left(\begin{bmatrix} 2 & 2 \\ -3 & -5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 2-\lambda & 2 \\ -3 & -5-\lambda \end{bmatrix} =$ $(2-\lambda)(-5-\lambda)-(-3)(2) = -10-2\lambda+5\lambda+\lambda^2+6 = -4+3\lambda+\lambda^2 = (-4-\lambda)(1-\lambda)$. As such, $\det(D-\lambda I) = 0$ if and only if $\lambda = 1, -4$, and so the eigenvalues of D are $\lambda = 1, -4$.

B2(e) To find the eigenvalues of $E = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix}$, we need to compute $\det(E-\lambda I) = \det\left(\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 3 & 5 \\ 0 & 2-\lambda & 7 \\ 0 & 0 & 3-\lambda \end{bmatrix} =$ $(1-\lambda)(2-\lambda)(3-\lambda)$ (since $E-\lambda I$ is a triangular matrix). As such, $\det(E-\lambda I) = 0$ if and only if $\lambda = 1, 2, 3$, and so the eigenvalues of E are $\lambda = 1, 2, 3$.

B2(f) To find the eigenvalues of $F = \begin{bmatrix} -4 & 6 & 6 \\ -2 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix}$, we need to compute

$$\det(F-\lambda I) = \det\left(\begin{bmatrix} -4 & 6 & 6 \\ -2 & 2 & 4 \\ -1 & 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -4-\lambda & 6 & 6 \\ -2 & 2-\lambda & 4 \\ -1 & 3 & 1-\lambda \end{bmatrix}.$$

I will compute this determinant by expanding along the first row:

$$\begin{aligned}
\det(F-\lambda I) &= (-4-\lambda)\begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} - 6\begin{vmatrix} -2 & 4 \\ -1 & 1-\lambda \end{vmatrix} + 6\begin{vmatrix} -2 & 2-\lambda \\ -1 & 3 \end{vmatrix} \\
&= (-4-\lambda)((2-\lambda)(1-\lambda)-12) - 6(-2(1-\lambda)+4) + 6(-6+(2-\lambda)) \\
&= (-4-\lambda)(2-2\lambda-\lambda+\lambda^2-12) - 6(-2+2\lambda+4) + 6(-6+2-\lambda) \\
&= (-4-\lambda)(-10-3\lambda+\lambda^2) - 6(2+2\lambda) + 6(-4-\lambda) \\
&= 40+12\lambda-4\lambda^2+10\lambda+3\lambda^2-\lambda^3-12-12\lambda-24-6\lambda \\
&= 4+4\lambda-\lambda^2-\lambda^3 \\
&= (-1-\lambda)(-2-\lambda)(2-\lambda)
\end{aligned}$$

(Note: To factor $4+4\lambda-\lambda^2-\lambda^3$, I first noticed that -1 was a solution to the polynomial. Factoring out the term $(-1-\lambda)$, I was left with $-4-\lambda^2$, which was easy to factor into $(-2-\lambda)(2-\lambda)$.)

As such, $\det(F-\lambda I) = 0$ if and only if $\lambda = -1, -2, 2$, and so the eigenvalues of F are $\lambda = -1, -2, 2$.