Solution to Practice 5a

B1 $A\vec{v}_1 = \begin{bmatrix} -3 & 1 & -1 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3-1 \\ 8+8 \\ 8+6 \end{bmatrix} = \begin{bmatrix} -4 \\ 16 \\ 14 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\vec{v}_1$ is not an eigenvector of A.

$A\vec{v}_2 = \begin{bmatrix} -3 & 1 & -1 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3+1 \\ 8-8 \\ 8-6 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so $\vec{v}_2$ is an eigenvector of A with corresponding eigenvalue -2.

$A\vec{v}_3 = \begin{bmatrix} -3 & 1 & -1 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+1 \\ 8-3 \\ 8-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 7 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so $\vec{v}_3$ is not an eigenvector of A.

$A\vec{v}_4 = \begin{bmatrix} -3 & 1 & -1 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3-1+1 \\ 8+3-8 \\ 8+1-6 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, so $\vec{v}_4$ is an eigenvector of A with corresponding eigenvalue -3.

$A\vec{v}_5 = \begin{bmatrix} -3 & 1 & -1 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3+1-2 \\ 8-3+16 \\ 8-1+12 \end{bmatrix} = \begin{bmatrix} -4 \\ 21 \\ 19 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, so $\vec{v}_5$ is not an eigenvector of A.

D1 Suppose that $\vec{v}$ is an eigenvector of A (with corresponding eigenvalue λ) and an eigenvector of B (with corresponding eigenvalue μ). Then $(A+B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda + \mu)\vec{v}$. So we see that $\vec{v}$ is an eigenvector of $A+B$, with corresponding eigenvalue $\lambda + \mu$. We also have $AB\vec{v} = A(B\vec{v}) = A(\mu\vec{v}) = \mu(A\vec{v}) = \mu(\lambda\vec{v}) = (\lambda\mu)\vec{v}$, so $\vec{v}$ is an eigenvector of AB with corresponding eigenvalue $\lambda\mu$.

D3 Suppose that $\vec{v}$ is an eigenvector of A, with corresponding eigenvalue λ, so that $A\vec{v} = \lambda\vec{v}$. This means that $A^{-1}(A\vec{v}) = A^{-1}(\lambda\vec{v})$, so $\vec{v} = A^{-1}(\lambda\vec{v}) = \lambda A^{-1}\vec{v}$. Now, since A is invertible, we know that $A\vec{x} = \vec{0}$ has only one solution, specifically $\vec{x} = \vec{0}$. As such, we know that 0 is not an eigenvalue of A. Even more specifically, this means that $\lambda \neq 0$. As such, we can divide by λ, and we have $(1/\lambda)\vec{v} = A^{-1}\vec{v}$. As such, we see that $\vec{v}$ is an eigenvector of A^{-1}, with corresponding eigenvalue $1/\lambda$.

D5 Suppose that A is an $n \times n$ matrix such that the sum of the entries in each row is c, and let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Then

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{bmatrix}$$

That is, the entries of $A\vec{v}$ are the sum of the entries of the corresponding rows of A. Since these rows all sum to c, we have

$$A\vec{v} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = c\vec{v}$$

As such, we see that $\vec{v}$ is an eigenvector of A.