Lecture 5e

Diagonalization

(pages 299-303)

The proof of the following fact lies outside the scope of this course, but we will need it to develop one of the main uses of eigenvectors, so we will state it without proof:

Fact: If A is an $n \times n$ matrix, and if $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A (that is, $\lambda_i \neq \lambda_j$ when $i \neq j$), and if all these eigenvalues have the property that their algebraic multiplicity equals their geometric multiplicity, then if we collect all the basis vectors for the eigenspaces of these eigenvalues, they form a basis for $\mathbb{R}^n$.

In the abstract, this may seem a bit confusing, but in practice what we are going to do is take all the basis vectors from the eigenspaces and make them the columns of a matrix P. The fact that these vectors form a basis for $\mathbb{R}^n$ tells us not only that P is an $n \times n$ matrix (as any basis of $\mathbb{R}^n$ must contain n vectors), but also that P is invertible, since the columns of P are linearly independent. And once we have P, here's what we do with it:

$$\begin{aligned} AP &= A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \\ &= \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= PD \end{aligned}$$

where D is an $n \times n$ diagonal matrix whose entries are the eigenvalues of A. (Note: we are no longer looking at the "distinct" eigenvalues of A, but instead are listing any eigenvalue the same number of times as its algebraic multiplicity.) Even more importantly, the non-zero entry in the i-th column of D is the eigenvalue corresponding to the eigenvector in the i-th column of P.

Combining our results, we have that

$$P^{-1}AP = D = \text{diag}(\lambda_1, \dots, \lambda_n)$$

<u>Definition:</u> If there exists an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, then we say A is **diagonalizable** and that the matrix P diagonalizes A to its diagonal form D.

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, as in our previous examples. Then we know that the eigenvalues for A are $\lambda = 2$ and $\lambda = 5$. Moreover, we saw that the eigenspace for $\lambda = 2$ is Span $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and the eigenspace for $\lambda = 5$ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Since the eigenvalues of A all have their algebraic multiplicity equal to their geometric multiplicity, we know that the matrix $P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ diagonalizes A to $D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$. We can verify this by computing $P^{-1}AP$. To do this, we first need to find P^{-1}, which I will do using the matrix inverse algorithm:

$$\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] \; R_1 \updownarrow R_2 \; \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array} \right] \; R_2 + 2R_1$$
$$\sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 2 \end{array} \right] \; (1/3)R_2 \; \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1/3 & 2/3 \end{array} \right] \; R_1 - R_2$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 1/3 & 2/3 \end{array} \right]$$

So we see that $P^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$. And now we compute $P^{-1}AP$ as follows:

$$\begin{aligned} P^{-1}(AP) &= \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \left(\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -4 & 5 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D \end{aligned}$$

It should be noted that this P is not the only matrix that diagonalizes A, nor is D the only diagonal form of A. First of all, we could switch the order of the eigenvectors used to make P, giving us $P_1 = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, and this would yield the diagonal form $D_1 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$. We could also replace any column of P (or P_1) with a scalar multiple of itself, as these would still be an eigenvector. As this eigenvector has the same eigenvalue as the first vector we used, this would not change D (or D_1). For example, the matrix $P_2 = \begin{bmatrix} 1 & 1 \\ -1/2 & 1 \end{bmatrix}$ diagonalizes A to $D_2 = D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, and the matrix $P_3 = \begin{bmatrix} 5 & 4 \\ 5 & -2 \end{bmatrix}$ diagonalizes A to diagonal form $D_3 = D_1 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$.

Example: Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$, as in our previous examples. Then the algebraic multiplicity of $\lambda = 3$ is 2, but the geometric multiplicity of $\lambda = 3$ is 1. As such, we can not find a basis of eigenvectors, and thus our technique for finding P to diagonalize B fails.

But can B be diagonalized? Again, the proof of this lies outside the scope of this course, but the following theorem and corollaries tell us that the answer is "no".

Theorem 6.2.2 (Diagonalization Theorem): An $n \times n$ matrix A can be diagonalized if and only if there exists a basis for $\mathbb{R}^n$ of eigenvectors of A. If such a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ exists, the matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ diagonalizes A to a diagonal matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i is the eigenvalue of A corresponding to $\vec{v}_i$ for $1 \leq i \leq n$.

Corollary 6.2.3: A matrix A is diagonalizable if and only if every eigenvalue of a A has its geometric multiplicity equal to its algebraic multiplicity.

Corollary 6.2.4: If an $n \times n$ matrix A has n distinct eigenvalues (that is, if all the eigenvalues of A have algebraic multiplicity equal to 1), then A is diagonalizable.

Example: Let $C = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{bmatrix}$. Then in the previous lecture, we saw that C had only one eigenvalue: $\lambda = 2$. And since the algebraic multiplicity of $\lambda = 2$ was 3, but the geometric multiplicity of $\lambda = 2$ was only 2, by Corollary 3 we know that C is not diagonalizable.

Example: Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then:

$$\begin{aligned} \det(M - \lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)(1-\lambda) - 4 \\ &= 1 - \lambda - \lambda + \lambda^2 - 4 \\ &= -3 - 2\lambda + \lambda^2 \\ &= (3-\lambda)(-1-\lambda) \end{aligned}$$

And so we see that the eigenvalues for M are $\lambda = 3$ and $\lambda = -1$. As M has distinct eigenvalues, we already know from Corollary 4 that M is diagonalizable.

In fact, we even know that its diagonal form is either $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, since we know the eigenvalues. What we don't know at this point is any matrix P that diagonalizes M. To find such a P, we will need to find the eigenspaces of the eigenvalues of M.

To find the eigenspace for $\lambda = 3$, we need to find the general solution to $(M - 3I)\vec{v} = \vec{0}$. To do this, we need to row reduce $M - 3I$ as follows:

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

And so our system is equivalent to the equation $v_1 - v_2 = 0$, or $v_1 = v_2$. Replacing the variable v_2 with the parameter s, we see that the general solution

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace corresponding to $\lambda = 3$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. To find the eigenspace for $\lambda = -1$, we need to find the general solution to $(M + 1I)\vec{v} = \vec{0}$. To do this, we need to row reduce $M + 1I$ as follows:

$$\begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{matrix} (1/2)R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

And so our system is equivalent to the equation $v_1 + v_2 = 0$, or $v_1 = -v_2$. Replacing the variable v_2 with the parameter s, we see that the general solution

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

Thus, the eigenspace corresponding to $\lambda = -1$ is $\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$. And now that we know the basis vectors for the eigenspaces, we know that $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizes M to diagonal form $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. (Other choices of P and D possible, of course.)

Before you begin work on the assignment, there are a few comments I want to add. First is the fact that not every polynomial can be factored. At least, not in the real numbers. Some of the problems in the textbook present this possibility–perhaps a 2×2 matrix doesn't have ANY eigenvalues, for example. In the problems I have assigned, all the characteristic polynomials factor completely over the integers, but I wanted to warn any of you who decide to do additional problems for practice. The next thing I wanted to point out is that for problems B2 and B3, you DO NOT need to find P^{-1} in order to answer these questions. (You do need to find P^{-1}, when applicable, for question B1.) Problems B2 and B3 should be solved more or less like this final example. (I say "more or less" because some are not diagonalizable!)

I would also like to note that I have omitted several theorems from sections 6.1, and all references to "similar matrices" from section 6.2. This was not by accident, and you are not expected or required to know these facts. Only the material from the lectures will be covered on the final exam.