Lecture 5c

Finding Eigenvectors

(pages 291-6)

Now that we know that eigenvalues are the solutions to $\det(A - \lambda I) = 0$, we can try to find the eigenvectors that have these eigenvalues. These vectors are the non-trivial solutions to the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$. That is, the eigenvectors with eigenvalue λ are the vectors in the solution space of the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$, minus the zero vector. As such, instead of looking for specific eigenvectors, it is easier to go ahead and add the zero vector and instead simply look for the solution space. This realization inspires the following definition:

Definition: Let λ be an eigenvalue of an $n \times n$ matrix A. Then the set containing the zero vector and all eigenvectors of A corresponding to λ is called the **eigenspace** of λ.

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, as in our previous examples. Then we have already seen that the eigenvalues of A are $\lambda = 2, 5$. To find the eigenspace corresponding to $\lambda = 2$. we need to find the general solution to the homogeneous system $(A - 2I)\vec{v} = \vec{0}$. To do this, we need to row reduce the coefficient matrix for this system, which is

$$A - 2I = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3-2 & 2 \\ 1 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

We row reduce $A - 2I$ as follows:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

So, our system is equivalent to the equation $v_1 + 2v_2 = 0$, or $v_1 = -2v_2$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

And so we see that the eigenspace corresponding to $\lambda = 2$ is $\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. Note that this is consistent with our original example, which showed that $\begin{bmatrix} 10 \\ -5 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2.

To find the eigenspace corresponding to $\lambda = 5$, we need to find the general solution to $(A-5I)\vec{v}=\vec{0}$. To do this, we need to row reduce the coefficient matrix for this system, which is

$$A-5I=\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}-\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}=\begin{bmatrix} 3-5 & 2 \\ 1 & 4-5 \end{bmatrix}=\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

We row reduce $A-5I$ as follows:

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{matrix} \\ R_2-R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, our system is equivalent to the equation $v_1-v_2=0$, or $v_1=v_2$. Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v}=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} s \\ s \end{bmatrix}=s\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

And so we see that the eigenspace corresponding to $\lambda=5$ is Span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Note that this is consistent with our original example, which showed that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A, with eigenvalue 5.

Another thing worth noting from our previous examples is that we looked at the matrix $A-\lambda I=\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$ as we were finding the eigenvalues for A. So, instead of computing $A-2I$ and $A-5I$ from scratch, we could instead plug our values for λ into this matrix. I'll use this technique in the next example.

Example: Let $B=\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$, as in our previous examples. Then we have already seen that the eigenvalues of B are $\lambda=3,-3,5$. To find the eigenspaces corresponding to these eigenvalues, we need to look at the solutions to the homogeneous systems $(B-\lambda I)\vec{v}=\vec{0}$. To do this, we will need to row reduce the coefficient matrix $B-\lambda I$. When finding the eigenvalues of B, we saw that

$$B-\lambda I=\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ -6 & 4-\lambda & 1 & 5 \\ 2 & 1 & 4-\lambda & -1 \\ 4 & 0 & 0 & -3-\lambda \end{bmatrix}$$

So, to find the eigenspace corresponding to $\lambda=3$, we need to row reduce $B-3I$ as follows:

$$B-3I=\begin{bmatrix} 3-3 & 0 & 0 & 0 \\ -6 & 4-3 & 1 & 5 \\ 2 & 1 & 4-3 & -1 \\ 4 & 0 & 0 & -3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 4 & 0 & 0 & -6 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_4 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 4 & 0 & 0 & -6 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1/4)R_1 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2+6R_1 \\ R_3-2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3-R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} (1/6)R_3 \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1+(3/2)R_3 \\ R_2+4R_3 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 &= 0 \\ v_2 + v_3 &= 0 \\ v_4 &= 0 \end{aligned}$$

If we replace v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

And so we see that the eigenspace corresponding to $\lambda = 3$ is Span $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find the eigenspace corresponding to $\lambda = -3$, we need to row reduce $B + 3I$ as follows:

$$B+3I=\begin{bmatrix} 3+3 & 0 & 0 & 0 \\ -6 & 4+3 & 1 & 5 \\ 2 & 1 & 4+3 & -1 \\ 4 & 0 & 0 & -3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} (1/6)R_1 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 + 6R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 5 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 7 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - 7R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & -48 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-1/48)R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 - 7R_3 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 &= 0 \\ v_2 + (3/4)v_4 &= 0 \\ v_3 - (1/4)v_4 &= 0 \end{aligned}$$

If we replace v_4 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ (-3/4)s \\ (1/4)s \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ -3/4 \\ 1/4 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -3/4 \\ 1/4 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 4 \end{bmatrix} \right\}$$

And so we see that the eigenspace corresponding to $\lambda = -3$ is Span $\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ 4 \end{bmatrix} \right\}$.

To find the eigenspace corresponding to $\lambda = 5$, we need to row reduce $B - 5I$ as follows:

$$B - 5I = \begin{bmatrix} 3-5 & 0 & 0 & 0 \\ -6 & 4-5 & 1 & 5 \\ 2 & 1 & 4-5 & -1 \\ 4 & 0 & 0 & -3-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} (-1/2)R_1 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} \\ R_2 + 6R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -8 \end{bmatrix} R_3 + R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix} R_4 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -R_2 \\ (1/4)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 + 5R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 &= 0 \\ v_2 - v_3 &= 0 \\ v_4 &= 0 \end{aligned}$$

If we replace v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

And so we see that the eigenspace corresponding to $\lambda = 5$ is Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.