Lecture 5b

Finding Eigenvalues

(pages 291-6)

We have been focusing our attention on eigenvectors, but before we can find eigenvectors, we will first need to find their eigenvalues. (Yes, without even knowing what the eigenvectors are!) To do this, let's consider what we are looking for. We know that λ is an eigenvalue for a matrix A if there is a non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

To solve this equation for λ, we can subtract $\lambda\vec{v}$ from both sides, getting

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

We can almost factor the vector $\vec{v}$ out of this equation, but first we need to change the scalar λ into an $n \times n$ matrix. "WHAT?" you ask. What I mean is that we want to notice that $\lambda\vec{v}$ is the same as $(\lambda I)\vec{v}$, where I is the $n \times n$ identity matrix. Using this, we see that

$$A\vec{v} = \lambda\vec{v} \text{ if and only if } (A - \lambda I)\vec{v} = \vec{0}$$

The equation $(A - \lambda I)\vec{v} = \vec{0}$ is a homogeneous system of linear equations with coefficient matrix $A - \lambda I$, and in this case we are looking for λ such that this system has a non-zero solution. That is, we want our homogeneous system to have non-trivial solutions. By the invertible matrix theorem, any system has a unique solution if and only if the determinant of the coefficient matrix is non-zero. Since we do NOT want our system to have a unique solution, this means that we want the determinant of $A - \lambda I$ to be zero. Summarizing, we have:

$$\lambda \text{ is an eigenvalue for } A \text{ if and only if } \det(A - \lambda I) = 0$$

Example: Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. In the previous lecture we saw that 2 and 5 were both eigenvalues for A. We see that these are the only eigenvalues for A by finding the eigenvalues for A as follows:

$$\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}\right) \\
&= (3-\lambda)(4-\lambda)-(1)(2) \\
&= 12-3\lambda-4\lambda+\lambda^2-2 \\
&= 10-7\lambda+\lambda^2 \\
&= (2-\lambda)(5-\lambda)
\end{aligned}$$

And so we have that $\det(A-\lambda I)=0$ if and only if $(2-\lambda)(5-\lambda)=0$. And so we see that λ is an eigenvalue for A if and only if $\lambda = 2, 5$.

Example: Consider the matrix $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$. In the previous lecture we saw that 3, -3, and 5 were eigenvalues for B. Are they the only eigenvalues? To find out, we need to calculate the eigenvalues for B as follows:

$$\begin{aligned}
\det(B-\lambda I) &= \det\left(\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ -6 & 4-\lambda & 1 & 5 \\ 2 & 1 & 4-\lambda & -1 \\ 4 & 0 & 0 & -3-\lambda \end{bmatrix}\right) \\
&= \text{(expanding along the first row) } (3-\lambda)\det\left(\begin{bmatrix} 4-\lambda & 1 & 5 \\ 1 & 4-\lambda & -1 \\ 0 & 0 & -3-\lambda \end{bmatrix}\right) \\
&= \text{(expanding along the third row) } (3-\lambda)(-3-\lambda)\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}\right) \\
&= (3-\lambda)(-3-\lambda)((4-\lambda)(4-\lambda)-(1)(1)) \\
&= (3-\lambda)(-3-\lambda)(16-8\lambda+\lambda^2-1) \\
&= (3-\lambda)(-3-\lambda)(15-8\lambda+\lambda^2) \\
&= (3-\lambda)(-3-\lambda)(3-\lambda)(5-\lambda) \\
&= (3-\lambda)^2(-3-\lambda)(5-\lambda)
\end{aligned}$$

And so we see that $\det(B-\lambda I)=0$ if and only if $(3-\lambda)^2(-3-\lambda)(5-\lambda)=0$. And thus we see that $\lambda = 3, -3, 5$ are the only eigenvalues for B.

These examples illustrate the fact that $\det(A-\lambda I)$ is a polynomial. This inspires the following definition:

Definition: Let A be an $n \times n$ matrix. Then $C(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of A.

And so, finding the eigenvalues of A is the same as finding the solutions to $C(\lambda) = 0$.

Example: In our examples, we found that the characteristic polynomial for $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ is $= (2-\lambda)(5-\lambda) = 10-7\lambda+\lambda^2$, and the characteristic polynomial for $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$ is $(3-\lambda)^2(-3-\lambda)(5-\lambda) = -135+72\lambda+6\lambda^2-8\lambda^3+\lambda^4$.
Because our only goal with the characteristic polynomial is to find its solutions, the factored form is preferable. For example, you'll note that I was able to find the eigenvalues for B without multiplying out all the term of its characteristic polynomial.