Lecture 5a

Eigenvalues and Eigenvectors

(pages 289-291)

We will finish off our course with a brief discussion of a curious creature called an eigenvector. We will not have time to do a thorough investigation of the uses of eigenvectors, but we will give you the tools you need to find them, so that you will be able to apply them to your future studies.

Definition: Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping. A non-zero vector $\vec{v} \in \mathbb{R}^n$ such that $L(\vec{v}) = \lambda\vec{v}$ (for some real number λ) is called an **eigenvector** of L, and the scalar λ is called an **eigenvalue** of L. The pair $\lambda, \vec{v}$ is called an **eigenpair**.

Before we look at some examples of eigenvectors, let's take a closer look at the definition. First of all, notice that L is in fact a linear operator (that is it has the same domain and codomain). This means that the standard matrix for L will be a square matrix. The next thing to notice is the requirement that eigenvectors be non-zero. This is simply because we know $L(\vec{0}) = \vec{0}$ for any L, so there is nothing at all special about the fact that there is some $\lambda \in \mathbb{R}$ such that $L(\vec{0}) = \lambda\vec{0}$. (In fact, all $\lambda \in \mathbb{R}$ have this property, but we will want there to be a unique λ for any eigenvector $\vec{v}$.) However, while the vector $\vec{0}$ can not be an eigenvector, the real number 0 *can* be an eigenvalue, if we find a non-zero vector $\vec{v}$ such that $L(\vec{v}) = \vec{0} = 0\vec{v}$.

Example: Let $\vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and consider $\text{proj}_{\vec{n}} : \mathbb{R}^2 \to \mathbb{R}^2$. Then, given any vector $\vec{m} \in \mathbb{R}^2$, we see that

$$\begin{aligned} \text{proj}_{\vec{n}}(\vec{m}) &= \frac{\vec{m}\cdot\vec{n}}{||\vec{n}||^2}\vec{n} \\ &= \frac{\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1^2+0^2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{m_1}{1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} m_1 \\ 0 \end{bmatrix} \end{aligned}$$

From this, we get that $\text{proj}_{\vec{n}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\text{proj}_{\vec{n}}$, with corresponding eigenvalue 1, since We also know that $\text{proj}_{\vec{n}}\left(\begin{bmatrix} m \\ 0 \end{bmatrix}\right) =$

$\begin{bmatrix} m \\ 0 \end{bmatrix}$ for any $m \in \mathbb{R}$, so the vectors $\begin{bmatrix} m \\ 0 \end{bmatrix}$ $(m \neq 0)$ are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 1.

We also have that $\text{proj}_{\vec{n}}\left(\begin{bmatrix} 0 \\ m \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $m \in \mathbb{R}$, so the vectors $\begin{bmatrix} 0 \\ m \end{bmatrix}$ $(m \neq 0)$ are also all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 0.

Lastly, we can look at he vectors $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$, with both $m_1, m_2 \neq 0$. Then $\text{proj}_{\vec{n}}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = \begin{bmatrix} m_1 \\ 0 \end{bmatrix}$. As $\begin{bmatrix} m_1 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ for any real number λ, we see that $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is not an eigenvector for $\text{proj}_{\vec{n}}$.

Note: The textbook generalizes this example to $\text{proj}_{\vec{n}}$ for any vector $\vec{n} \in \mathbb{R}^n$. The result in the general case is that $\text{proj}_{\vec{n}}(s\vec{n}) = s\vec{n}$, so the vectors $s\vec{n}$ (for $s \neq 0$) are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 1. This corresponds to looking at the vectors $\begin{bmatrix} m \\ 0 \end{bmatrix} = m \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in our example. And, in general, if $\vec{v}$ is orthogonal to $\vec{n}$, then $\text{proj}_{\vec{n}}(s\vec{v}) = \vec{0}$, so the vectors $s\vec{v}$ (for $s \neq 0$) are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 0. (This corresponds to the vectors $\begin{bmatrix} 0 \\ m \end{bmatrix}$ in our example.) And only vectors that are either multiples of $\vec{n}$ or orthogonal to $\vec{n}$ are eigenvectors of $\text{proj}_{\vec{n}}$.

Example: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping for a dilation by a factor of 5. That is, L is defined by $L(\vec{v}) = 5\vec{v}$. Then every vector $\vec{v} \neq \vec{0}$ is an eigenvector of L, with corresponding eigenvalue 5.

Example: Consider $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, where θ is NOT an integer multiple of π. Well, $\vec{w}$ is a scalar multiple of $\vec{v}$ if and only if they point in the same direction or opposite direction. But a rotation by our θ will not preserve direction. Thus, we know that $R_\theta(\vec{v}) \neq s\vec{v}$ for any $s \in \mathbb{R}$, and so there are no eigenvectors of R_θ.

Now, usually when we are dealing with linear mappings, we use their standard matrix. As such, we will make use of the following "matrix" definition for eigenvectors as well.

Definition: Suppose that A is an $n \times n$ matrix. A non-zero vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$ is called an **eigenvector** of A, and the scalar λ is called an **eigenvalue** of A. The pair $\lambda, \vec{v}$ is called an **eigenpair**.

Example: Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Then we see that:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A, with corresponding eigenvalue 5. We also have:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 30-10 \\ 10-20 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix} = 2\begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

so $\begin{bmatrix} 10 \\ -5 \end{bmatrix}$ is an eigenvector for A, with corresponding eigenvalue 2. Finally, we look at:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not an eigenvector for A.

Example: Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$. Then we see that:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+0+0+0 \\ -6+8-1-10 \\ 2+2-4+2 \\ 4+0+0+6 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 2 \\ 10 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$ is not an eigenvector for B. But

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+4+1+0 \\ 0+1+4+0 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue 5, and

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+20-5+0 \\ 0+5-20+0 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ -15 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue 3, and

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0-24+2+40 \\ 0-6+8-8 \\ 0+0+0-24 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ -6 \\ -24 \end{bmatrix} = -3 \begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue -3.