Solution to Practice 4l

B1(a) Area$(\vec{u},\vec{v}) = \left|\det \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}\right| = |12-14| = |-2| = 2.$

B1(b)

$$A\vec{u} = \begin{bmatrix} 2 & 8 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 6+56 \\ -3+35 \end{bmatrix} = \begin{bmatrix} 62 \\ 32 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 2 & 8 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4+32 \\ -2+20 \end{bmatrix} = \begin{bmatrix} 36 \\ 18 \end{bmatrix}$$

B1(c) $\det A = 10 + 8 = 18$

B1(d) Area$(A\vec{u}, A\vec{v}) = \left|\det \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\right| = \left|\det \begin{bmatrix} 62 & 36 \\ 32 & 18 \end{bmatrix}\right| = |1116 - 1152| = |-36| = 36.$

—or—

Area$(A\vec{u}, A\vec{v}) = |\det A|$Area$(\vec{u},\vec{v}) = 18 \cdot 2 = 36.$

B2

$$H(\vec{e}_1) = A\vec{e}_1 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H(\vec{e}_2) = A\vec{e}_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

So Area$(H(\vec{e}_1), H(\vec{e}_2)) = \left|\det \begin{bmatrix} H(\vec{e}_1) & H(\vec{e}_2) \end{bmatrix}\right| = \left|\det \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right| = |1 - 0| = 1.$

B3(a) Let $B = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -3 & 5 \\ 0 & -1 & 3 \end{bmatrix}$. Then Volume$(\vec{u},\vec{v},\vec{w}) = |\det B|$. And

$$\begin{aligned} \det B \quad & \text{(expanding along the first row)} \\ &= 2\begin{vmatrix} -3 & 5 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 5 \\ 0 & 3 \end{vmatrix} + 0 \\ &= 2(-9+5) - (-3+0) = -8 + 3 = -5 \end{aligned}$$

So Volume$(\vec{u},\vec{v},\vec{w}) = |-5| = 5.$

B3(b)

$$\begin{aligned}
\det A \quad & \text{(expanding along the third row)} \\
&= -\begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix} + 0 + 0 \\
&= -(0 - 15) = 15
\end{aligned}$$

B3(c) Volume$(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|$Volume$(\vec{u}, \vec{v}, \vec{w}) = 15 \cdot 5 = 75$.

B4(a) Let $B = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ -4 & 3 & 5 \end{bmatrix}$. Then Volume$(\vec{u}, \vec{v}, \vec{w}) =$ $|\det B|$. To compute $\det B$, let's first row reduce B:

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ -4 & 3 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 11 & 1 \end{bmatrix}.$$

As these row operations do not affect the determinant, we have that

$$\begin{aligned}
\det B \quad &= \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ -4 & 3 & 5 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 11 & 1 \end{vmatrix} \\
&= \text{(expanding along the first column)} \begin{vmatrix} -3 & 4 \\ 11 & 1 \end{vmatrix} + 0 + 0 \\
&= -3 - 44 = -47
\end{aligned}$$

And so we have Volume$(\vec{u}, \vec{v}, \vec{w}) = |\det B| = |-47| = 47$.

B4(b)

$$\begin{aligned}
\det A \quad & \text{(expanding along the first column)} \\
&= 4\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} + 0 - \begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} \\
&= 4(2 - 12) - (-6 - 1) = -40 + 7 = -33
\end{aligned}$$

B4(c) Volume$(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|$Volume$(\vec{u}, \vec{v}, \vec{w}) = 33 \cdot 47 = 1551$.

B5(a) Let $B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 5 \\ 1 & 3 & 0 & 7 \end{bmatrix}$. Then Volume$(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) =$

$|\det B|$. To compute $\det B$, let's first row reduce B:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 5 \\ 1 & 3 & 0 & 7 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 2 & -1 & 6 \end{bmatrix} R_2 \updownarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & 6 \end{bmatrix} R_4 + 2R_2 \qquad \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 12 \end{bmatrix} R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 13 \end{bmatrix}.$$

Thus

$$\begin{aligned} \det B &= -\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 13 \end{bmatrix} \\ &= -(1)(-1)(1)(13) = 13 \end{aligned}$$

And so we see that $\text{Volume}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = 13$.

B5(b) $\text{Volume}(A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4) = |\det A| \text{Volume}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, so we need to compute $\det A$. First, I use row operations to notice that

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & 3 & 7 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & 3 & 7 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}.$$

As this row operation does not change the determinant, we have the following:

$$\det A = \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ -1 & 3 & 7 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

(expanding along the first row)

$$= 0 + 0 + 0 + (-1) \begin{vmatrix} 2 & 3 & 1 \\ -1 & 3 & 7 \\ 0 & 2 & 1 \end{vmatrix}$$

(expanding the submatrix along the first column)

$$= -\left(2 \begin{vmatrix} 3 & 7 \\ 2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 0\right)$$

$$= -2(3 - 14) - (3 - 2) = 22 - 1 = 21$$

And so we have that $\text{Volume}(A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4) = 21 \cdot 13 = 273.$