Solution to Practice 4j

B1(a) In the previous assignment, we found that the cofactor matrix is $\begin{bmatrix} 2 & -7 \\ -4 & -3 \end{bmatrix}$. And since the determinant is $-6 - 28 - 34$, we have that the inverse is

$$-\frac{1}{34}\begin{bmatrix} 2 & -4 \\ -7 & -3 \end{bmatrix}$$

B1(b) In the previous assignment, we found that the cofactor matrix is $\begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$. And since the determinant is $15 + 8 = 23$, we have that the inverse is

$$\frac{1}{23}\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$$

B1(c) In the previous assignment, we found that the cofactor matrix is $\begin{bmatrix} 12 & 12 & -21 \\ -23 & 6 & 4 \\ 9 & 9 & 6 \end{bmatrix}$. We can expand along the first row to get that the determinant is $0 + 2(12) - 3(-21) = 87$. Thus, the inverse is

$$\frac{1}{87}\begin{bmatrix} 12 & -23 & 9 \\ 12 & 6 & 9 \\ -21 & 4 & 6 \end{bmatrix}$$

B1(d) In the previous assignment, we found that the cofactor matrix is $\begin{bmatrix} 1 & 2 & -1 \\ -2 & -5 & 1 \\ -2 & -4 & 1 \end{bmatrix}$. We can expand along the first row to get that the determinant is $1(1) + 0(2) + 2(-1) = -1$. Thus, the inverse is

$$-\begin{bmatrix} 1 & -2 & -2 \\ 2 & -5 & -4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 5 & 4 \\ 1 & -1 & -1 \end{bmatrix}$$

B1(e) In the previous assignment, we found that the cofactor matrix is $\begin{bmatrix} -6 & 0 & 0 \\ -18 & 12 & 42 \\ -12 & 12 & 24 \end{bmatrix}$. We can expand along the first column to get that the determinant is $6(-6) + 0 + 0 = -36$. Thus, the inverse is

$$-\frac{1}{36}\begin{bmatrix} -6 & -18 & -12 \\ 0 & 12 & 12 \\ 0 & 42 & 24 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -2 \\ 0 & -7 & -4 \end{bmatrix}$$

B2(a) First we need to compute all the cofactors:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} t & -1 \\ 1 & 4 \end{vmatrix} = 4t + 1$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} = -(12 - 2) = -10$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & t \\ -2 & 1 \end{vmatrix} = 3 + 2t = 2t + 3$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4 - 3) = -1$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} = 8 + 6 = 14$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = -(2 + 2) = -4$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 3 \\ t & -1 \end{vmatrix} = -1 - 3t = -3t - 1$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2 - 9) = 11$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & t \end{vmatrix} = 2t - 3$$

So $\text{cof } A = \begin{bmatrix} 4t+1 & -10 & 2t+3 \\ -1 & 14 & -4 \\ -3t-1 & 11 & 2t-3 \end{bmatrix}$.

B2(b)

$$\begin{aligned}
A(\text{cof } A)^T &= \begin{bmatrix} 2 & 1 & 3 \\ 3 & t & -1 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 4t+1 & -1 & -3t-1 \\ -10 & 14 & 11 \\ 2t+3 & -4 & 2t-3 \end{bmatrix} \\
&= \begin{bmatrix} 8t+2-10+6t+9 & -2+14-12 & -6t-2+11+6t-9 \\ 12t+3-10t-2t-3 & -3+14t+4 & -9t-3+11t-2t+3 \\ -8t-2-10+8t+12 & 2+14-16 & 6t+2+11+8t-12 \end{bmatrix} \\
&= \begin{bmatrix} 14t+1 & 0 & 0 \\ 0 & 14t+1 & 0 \\ 0 & 0 & 14t+1 \end{bmatrix}
\end{aligned}$$

And since $A(\text{cof A})^T = (\det A)I$, we see that $\det A = 14t + 1$.

When $14t+1 \neq 0$, (i.e. when $t \neq -1/14$), we get that $A^{-1} = \frac{1}{\det A}(\text{cof } A)^T$, so

$$A^{-1} = \frac{1}{14t+1}\begin{bmatrix} 4t+1 & -1 & -3t-1 \\ -10 & 14 & 11 \\ 2t+3 & -4 & 2t-3 \end{bmatrix}$$

D2

$(A^{-1})_{23} = \left(\frac{1}{\det A}(\text{cof } A)^T\right)_{23} = \frac{1}{\det A}((\text{cof } A)^T)_{23}$. So, we need to find det A, and we need to find $((\text{cof } A)^T)_{23}$.

Let's compute det A first: Expanding on the first column, we get that $\det A = 2\begin{vmatrix} -1 & 3 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 3 \end{vmatrix}$. Expanding this submatrix along the second row, we get $\det A = 2\left(-1\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix}\right) = -2(9-0) = -18.$

Now, $((\text{cof } A)^T)_{23} = (\text{cof A})_{32} = C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{vmatrix} =$ (expanding along the first column) $-\left(2\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix}\right) = -2(9-0) = -18.$

So $(A^{-1})_{23} = (-1/18)(-18) = 1.$

We also have $((\text{cof } A)^T)_{42} = (\text{cof A})_{24} = C_{24} = (-1)^{2+4}\begin{vmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = 0.$

This means that $(A^{-1})_{42} = \left(\frac{1}{\det A}(\text{cof } A)^T\right)_{42} = \frac{1}{\det A}((\text{cof } A)^T)_{42} = (-1/18)(0) = 0.$