Lecture 4k

Cramer's Rule

(pages 276-8)

While the cofactor method isn't the most practical way to compute the inverse of a matrix, it does give us the useful formula that

$$A^{-1} = \frac{1}{\det A}(\text{cof } A)^T$$

And we can now apply this formula to a system of equations. Because, in the case when A is invertible, we know that the solution to the system $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$. But this means that $\vec{x} = \frac{1}{\det A}(\text{cof } A)^T\vec{b}$. If we recall that matrix multiplication is the dot product of a row with a column, and that the rows of $(\text{cof } A)^T$ are the columns of cof A, then we see that

$$x_i = \frac{1}{\det A}(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni})$$

The equation $b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}$ looks like the calculation for a determinant expanded along the i-th column. In fact, this is the determinant of the matrix N_i that can be obtained from A by replacing the i-th column with $\vec{b}$, since the cofactors of column i do not involve column i, and we would have $a_{ki} = b_k$ for all k.

Thus, we have

$$x_i = \frac{\det N_i}{\det A}$$

This expression for the solution to a system of equations is known as **Cramer's Rule**.

Example: Let's use Cramer's Rule to solve this system of equations:

$$\begin{array}{rrcr} 4x_1 & +7x_2 & = & 3 \\ -5x_1 & +3x_2 & = & -8 \end{array}$$

We note that our coefficient matrix is $A = \begin{bmatrix} 4 & 7 \\ -5 & 3 \end{bmatrix}$, and that $\vec{b} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$. We get the matrix N_1 by replacing the first column of A with $\vec{b}$, so $N_1 = \begin{bmatrix} 3 & 7 \\ -8 & 3 \end{bmatrix}$. We get the matrix N_2 by replacing the second column of A with

$\vec{b}$, so $N_2 = \begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix}$. Now that we have the matrices, we need to compute their determinants:

$\det A = \det \begin{bmatrix} 4 & 7 \\ -5 & 3 \end{bmatrix} = 12 + 35 = 47$

$\det N_1 = \det \begin{bmatrix} 3 & 7 \\ -8 & 3 \end{bmatrix} = 9 + 56 = 65$

$\det N_2 = \det \begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix} = -32 + 15 = -17$

This means that

$$x_1 = \frac{\det N_1}{\det A} = \frac{65}{47} \quad \text{and} \quad x_2 = \frac{\det N_2}{\det A} = -\frac{17}{47}$$

We can verify that this is the solution by plugging it into our system:

$$\begin{gathered} 4(65/47) + 7(17/47) = (260 - 119)/47 = 141/47 = 3 \\ -5(65/47) + 3(17/47) = (-325 - 51)/47 = -376/47 = -8 \end{gathered}$$

Cramer's rule seems nice in this 2×2 case, and it cleverly takes care of the fractions for us. But as we'll see in the next example, Cramer's rule quickly becomes time consuming.

Example: To use Cramer's Rule to solve the system of equations

$$\begin{array}{rrrcl} 7x_1 & +2x_2 & -3x_3 & = & 6 \\ 4x_1 & +7x_2 & +6x_3 & = & 5 \\ 8x_1 & -9x_2 & -5x_3 & = & 4 \end{array}$$

we first find that $A = \begin{bmatrix} 7 & 2 & -3 \\ 4 & 7 & 6 \\ 8 & -9 & -5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$. Then we use this information to find N_1, N_2, and N_3. We get N_1 by replacing the first column of A with $\vec{b}$, so

$N_1 = \begin{bmatrix} 6 & 2 & -3 \\ 5 & 7 & 6 \\ 4 & -9 & -5 \end{bmatrix}$

We get N_2 by replacing the second column of A with $\vec{b}$, so

$N_2 = \begin{bmatrix} 7 & 6 & -3 \\ 4 & 5 & 6 \\ 8 & 4 & -5 \end{bmatrix}$

We get N_3 by replacing the third column of A with $\vec{b}$, so

$$N_3 = \begin{bmatrix} 7 & 2 & 6 \\ 4 & 7 & 5 \\ 8 & -9 & 4 \end{bmatrix}$$

Next we need to compute the determinants of A, N_1, N_2, and N_3:

$$\det A = \det \begin{bmatrix} 7 & 2 & -3 \\ 4 & 7 & 6 \\ 8 & -9 & -5 \end{bmatrix} = 7\begin{vmatrix} 7 & 6 \\ -9 & -5 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & 7 \\ 8 & -9 \end{vmatrix}$$

$= 7(-35+54) - 2(-20-48) - 3(-36-56) = 7(19) - 2(-68) - 3(-92) = 133 + 136 + 276$

$= 545$

$$\det N_1 = \det \begin{bmatrix} 6 & 2 & -3 \\ 5 & 7 & 6 \\ 4 & -9 & -5 \end{bmatrix} = 6\begin{vmatrix} 7 & 6 \\ -9 & -5 \end{vmatrix} - 2\begin{vmatrix} 5 & 6 \\ 4 & -5 \end{vmatrix} - 3\begin{vmatrix} 5 & 7 \\ 4 & -9 \end{vmatrix}$$

$= 6(-35+54) - 2(-25-24) - 3(-45-28) = 6(19) - 2(-49) - 3(-73) = 114 + 98 + 219$

$= 431$

$$\det N_2 = \det \begin{bmatrix} 7 & 6 & -3 \\ 4 & 5 & 6 \\ 8 & 4 & -5 \end{bmatrix} = 7\begin{vmatrix} 5 & 6 \\ 4 & -5 \end{vmatrix} - 6\begin{vmatrix} 4 & 6 \\ 8 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 8 & 4 \end{vmatrix}$$

$7(-25-24) - 6(-20-48) - 3(16-40) = 7(-49) - 6(-68) - 3(-24) = -33 + 408 + 72$

$= 137$

$$\det N_3 = \det \begin{bmatrix} 7 & 2 & 6 \\ 4 & 7 & 5 \\ 8 & -9 & 4 \end{bmatrix} = 7\begin{vmatrix} 7 & 5 \\ -9 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 8 & 4 \end{vmatrix} + 6\begin{vmatrix} 4 & 7 \\ 8 & -9 \end{vmatrix}$$

$= 7(28+45) - 2(16-40) + 6(-36-56) = 7(73) - 2(-24) + 6(-92) = 511 + 48 - 552$

$= 7$

And so we see that the solution to the system is

$$x_1 = \frac{\det N_1}{\det A} = \frac{431}{545}, \quad x_2 = \frac{\det N_2}{\det A} = \frac{137}{545}, \quad x_3 = \frac{\det N_3}{\det A} = \frac{7}{545}$$