Lecture 4i

The Cofactor Matrix

(pages 274-5)

As we continue our study of determinants, we will want to make use of the following matrix:

Definition: Let A be an $n \times n$ matrix. We define the **cofactor matrix** of A, denoted cof A, by

$$(\text{cof } A)_{ij} = C_{ij}$$

That is, the ij entry of cof A is the cofactor of the ij entry of A.

Note: We have not yet looked at the cofactors of a 2×2 matrix, as the determinant of a 2×2 matrix was given by a formula. But now we want to be able to find the cofactor matrix of a 2×2 matrix, and in doing so, we see that we need to define the determinant of a 1×1 matrix. Perhaps it is due to its simplicity that the textbook never explicitly states that the determinant of a 1×1 matrix $[a]$ is a. Note that this is consistent with Theorem 5.1.3, and with the definition of the determinant of a 2×2 matrix being the same as expanding by cofactors.

Example: To determine the cofactor matrix of $A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$, we first need to compute all the cofactors:

$C_{11} = (-1)^{1+1}\det\begin{bmatrix} -7 \end{bmatrix} = -7 \quad C_{12} = (-1)^{1+2}\det\begin{bmatrix} 4 \end{bmatrix} = -4$

$C_{21} = (-1)^{2+1}\det\begin{bmatrix} 3 \end{bmatrix} = -3 \quad C_{22} = (-1)^{2+2}\det\begin{bmatrix} 2 \end{bmatrix} = 2$

Note that I used the notation $\det\begin{bmatrix} -7 \end{bmatrix}$ instead of the abbreviated notation $\begin{vmatrix} -7 \end{vmatrix}$ in this case, to prevent confusion with the idea of the absolute value.

And so we have

$$\text{cof } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ -3 & 2 \end{bmatrix}$$

Example: To determine the cofactor matrix of $B = \begin{bmatrix} 7 & 1 & 3 \\ 4 & -2 & -5 \\ 9 & 8 & -3 \end{bmatrix}$, we first need to compute all the cofactors:

$C_{11} = (-1)^{1+1}\begin{vmatrix} -2 & -5 \\ 8 & -3 \end{vmatrix} = 6 + 40 = 46$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & -5 \\ 9 & -3 \end{vmatrix} = -(-12+45) = -33$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & -2 \\ 9 & 8 \end{vmatrix} = 32+18 = 50$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 8 & -3 \end{vmatrix} = -(-3-24) = 27$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 7 & 3 \\ 9 & -3 \end{vmatrix} = -21-27 = -48$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 7 & 1 \\ 9 & 8 \end{vmatrix} = -(56-9) = -47$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} = -5+6 = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 7 & 3 \\ 4 & -5 \end{vmatrix} = -(-35-12) = 47$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 7 & 1 \\ 4 & -2 \end{vmatrix} = -14-4 = -18$$

And so we have

$$\text{cof } B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 46 & -33 & 50 \\ 27 & -48 & -47 \\ 1 & 47 & -18 \end{bmatrix}$$