Solution to Practice 4h

**B4(a)** $AB = \begin{bmatrix} -21 & 2 \\ 17 & 1 \end{bmatrix}$

$\det A = (4)(2) - (-3)(-1) = 5$

$\det B = (-5)(2) - (1)(1) = -11$

$\det AB = (-21)(1) - (17)(2) = -55 = 5(-11) = (\det A)(\det B)$

**B4(b)** $AB = \begin{bmatrix} 8 & 3 & 5 \\ -3 & 2 & 19 \\ 4 & 2 & 4 \end{bmatrix}$

$\det A =$ (expanding along the third row) $0 + (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + (-1)^{3+3} \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -(12+1) + (-9-1) = -13 - 10 = -23$

$\det B =$ (expanding along the second row) $3(-1)^{2+1} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + (-1)^{2+2} \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 0 = -3(4-3) + (8-3) = -3 + 5 = 2$

$\det AB =$ (expanding along the first row) $8(-1)^{1+1} \begin{vmatrix} 2 & 19 \\ 2 & 4 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} -3 & 19 \\ 4 & 4 \end{vmatrix} + 5(-1)^{1+3} \begin{vmatrix} -3 & 2 \\ 4 & 2 \end{vmatrix} = 8(8-38) - 3(-12-76) + 5(-6-8) = -240 + 264 - 70 = -46 = (-23)(2) = (\det A)(\det B)$

**D2** Let $\det A = a$ and $\det B = b$. Then $\det AB = (\det A)(\det B) = ab$, so $\det AB = 0$ if and only if $ab = 0$. But we know that the product of two real numbers is zero if and only if at least one of them is zero, so $ab = 0$ if and only if $a = 0$ or $b = 0$. Thus, $\det AB = 0$ if and only if $\det A = 0$ or $\det B = 0$.

Sideproof: (Not necessary, but in case you haven't seen it before.) Let $a$ and $b$ be real numbers, and suppose that $ab = 0$. If $a = 0$, then $ab = 0$. If $a \neq 0$, then we can divide both sides by $a$, and get $b = 0$. Thus, either $a = 0$ or $b = 0$. On the other hand, if $a = 0$ or $b = 0$, then we have $ab = 0$. So we know that $ab = 0$ if and only if $a = 0$ or $b = 0$.

**D4** Let $A$ be an $n \times n$ matrix, and let $P$ be an invertible $n \times n$ matrix. Then

$$\begin{aligned} \det(P^{-1}AP) &= (\det P^{-1})(\det A)(\det P) \\ &= (\det A)(\det P^{-1})(\det P) \\ &= \det AP^{-1}P \\ &= \det AI \\ &= \det A \end{aligned}$$

Note: The key to this proof is the fact that multiplication of real numbers, such as determinants, *is* commutative.