Solution to Practice 4f

B1(a) $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 2 & -1 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & -3 & -7 \\ 0 & 1 & -6 \end{bmatrix} \; R_2 \updownarrow R_3$

$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & -3 & -7 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -25 \end{bmatrix} = B.$

Then $\det B = (1)(1)(-25) = -25$. Number of row swaps $= 2$, which is even, so we do not need to alter $\det B$. And we didn't multiply any rows by a scalar, so we see that $\det A = -25$.

B1(b) $A = \begin{bmatrix} 1 & 4 & -4 \\ 2 & -3 & 3 \\ -3 & -4 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & -4 \\ 0 & -11 & 11 \\ 0 & 8 & -8 \end{bmatrix} \begin{matrix} \\ \\ R_3 + (8/11)R_2 \end{matrix}$

$\sim \begin{bmatrix} 1 & 4 & -4 \\ 0 & -11 & 11 \\ 0 & 0 & 0 \end{bmatrix} = B.$

Then $\det B = 0$, and therefore $\det A = 0$.

B1(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 1 \\ 3 & 5 & -6 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -5 \\ 0 & -1 & -15 \end{bmatrix} \; R_2 \updownarrow R_3$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -15 \\ 0 & -8 & -5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 8R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -15 \\ 0 & 0 & 115 \end{bmatrix} = B.$

Then $\det B = (1)(-1)(115) = -115$. We performed one row swap, so we need to multiply $\det B$ by (-1), but we did not multiply any rows by a scalar, so this is the only thing we need to multiply $\det B$ by. And so we have $\det A = -\det B = 115$.

B1(d) $A = \begin{bmatrix} 7 & 1 & -1 & 1 \\ 3 & 3 & -4 & 5 \\ 3 & 2 & 1 & 4 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_4 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 3 & -4 & 5 \\ 3 & 2 & 1 & 4 \\ 7 & 1 & -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \\ R_4 - 7R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & -1 & 4 & 1 \\ 0 & -6 & 6 & -6 \end{bmatrix} \; R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & -6 & 6 & -6 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 6R_2 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -18 & -12 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 18R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -48 \end{bmatrix} = B.$

Then $\det B = (1)(-1)(-1)(-48) = -48$. We performed two row swaps, and did not multiply any rows by a scalar, so $\det A = \det B = -48$.

B1(e) $A = \begin{bmatrix} 1 & 10 & 7 & -9 \\ 7 & -7 & 7 & 7 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} \\ (1/7)R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 10 & 7 & -9 \\ 1 & -1 & 1 & 1 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{matrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 10 & 7 & -9 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 11 & 6 & -10 \\ 0 & 0 & 4 & 0 \\ 0 & -6 & 7 & 4 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + (6/11)R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 11 & 6 & -10 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 113/11 & -16/11 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - (113/44)R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 11 & 6 & -10 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -16/11 \end{bmatrix} = B$$

Then $\det B = (1)(11)(4)(-16/11) = -64$. We performed 1 row swap, and we multiplied one row by (1/7). So we have $\det A = (-1)(7)\det B = 448$.

—–OR—–

I tried a couple of different row reductions, and always ended up needing fractions. Then it occurred to me that instead of making B upper-triangular (i.e. in row echelon form), I could make B lower-triangular. Here is that solution:

$$A = \begin{bmatrix} 1 & 10 & 7 & -9 \\ 7 & -7 & 7 & 7 \\ 2 & -2 & 6 & 2 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 + 9R_4 \\ R_2 - 7R_4 \\ R_3 - 2R_4 \\ \\ \end{matrix} \sim \begin{bmatrix} -26 & -17 & 43 & 0 \\ 28 & 14 & -21 & 0 \\ 8 & 4 & -2 & 0 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \\ (-1/2)R_3 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} -26 & -17 & 43 & 0 \\ 28 & 14 & -21 & 0 \\ -4 & -2 & 1 & 0 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 - 43R_2 \\ R_2 + 21R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 146 & 69 & 0 & 0 \\ -56 & -28 & 0 & 0 \\ -4 & -2 & -1 & 0 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} \\ (-1/28)R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 146 & 69 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & -1 & 0 \\ -3 & -3 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 - 69R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 8 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & -1 & 0 \\ -3 & -3 & 4 & 1 \end{bmatrix} = B$$

Then $\det B = (8)(1)(1)(1) = 8$. We did not perform any row swaps, but we did multiply a row by -1/2 and another row by -1/28. Thus, $\det A = (-2)(-28)\det B = 448$.

B1(f) $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 1 & 2 & 1 & -1 \\ -4 & 5 & 8 & -3 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 7 & 4 & 3 \\ 0 & -1 & 0 & -2 \\ 0 & 17 & 12 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \updownarrow R_3 \\ \\ \\ \end{matrix}$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 7 & 4 & 3 \\ 0 & 17 & 12 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 7R_2 \\ R_4 + 17R_2 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 4 & -11 \\ 0 & 0 & 12 & -33 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 4 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Then $\det B = 0$, so $\det A = 0$.

B2(a) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & -2 & 2 \\ -1 & 4 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -1 \\ 0 & 7 & 8 \end{bmatrix} = B.$

Then $\det A = \det B =$ (expanding along the first column) $(1)(-1)^{1+1} \begin{vmatrix} -5 & -1 \\ 7 & 8 \end{vmatrix} + 0 + 0 = (-5)(8) - (7)(-1) = -33.$

B2(b) $A = \begin{bmatrix} 6 & 1 & 2 \\ -4 & -1 & 3 \\ -3 & 1 & 5 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 6 & 1 & 2 \\ 2 & 0 & 5 \\ -9 & 0 & 3 \end{bmatrix} = B.$

Then $\det A = \det B =$ (expanding along the second column) $(1)(-1)^{1+2} \begin{vmatrix} 2 & 5 \\ -9 & 3 \end{vmatrix} + 0 + 0 = -((2)(3) - (-9)(5)) = -51.$

B2(c) $A = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 1 & 3 & -1 & -1 \\ 2 & 2 & -2 & 7 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix}$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = B.$$

Then $\det A = \det B = (1)(2)(2)(-4) = -16$. (I realize the instructions said to "use a combination...", but I couldn't imagine an easier way to calculate this determinant. Perhaps I've grown too accustomed to doing elementary row operations...)

B2(d) $A = \begin{bmatrix} 2 & 3 & 2 & -2 \\ 3 & 1 & 4 & 1 \\ 1 & 2 & 4 & 4 \\ 2 & -1 & 5 & 6 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ R_4 - 2R_3 \end{matrix} \sim \begin{bmatrix} 0 & -1 & -6 & -10 \\ 0 & -5 & -8 & -11 \\ 1 & 2 & 4 & 4 \\ 0 & -5 & -3 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 5R_1 \\ \\ R_4 - 5R_1 \end{matrix}$

$$\sim \begin{bmatrix} 0 & -1 & -6 & -10 \\ 0 & 0 & 22 & 39 \\ 1 & 2 & 4 & 4 \\ 0 & 0 & 27 & 48 \end{bmatrix} = B.$$

Then $\det A = \det B =$ (expanding along the first column) $0+0+(1)(-1)^{3+1} \begin{vmatrix} -1 & -6 & -10 \\ 0 & 22 & 39 \\ 0 & 27 & 48 \end{vmatrix} +$

$$0 = \begin{vmatrix} -1 & -6 & -10 \\ 0 & 22 & 39 \\ 0 & 27 & 48 \end{vmatrix} = \text{ (expanding along the first column) } (-1)(-1)^{1+1} \begin{vmatrix} 22 & 39 \\ 27 & 48 \end{vmatrix} + 0 + 0 = -((22)(48) - (27)(39)) = -3.$$