Solution to Practice 4d

B1(d) Because the second row is all zeros, the determinant of this matrix is 0.

B1(e) Because this is a lower-triangular matrix, the determinant is the product of the diagonal entries: $(3)(2)(1) = 6$.

B1(f) Because this is an upper-triangular matrix, the determinant is the product of the diagonal entries: $(4)(3)(2)(1) = 24$.

B3(a) Expanding along the third row, we get that $\begin{vmatrix} 2 & 5 & 4 \\ -3 & 2 & 1 \\ 3 & -6 & 0 \end{vmatrix} =$

$(3)(-1)^{3+1} \begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix} + (-6)(-1)^{3+2} \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} + 0 =$

$3((5)(1) - (2)(4)) + 6((2)(1) - (-3)(4)) = -9 + 84 = 75$

B3(b) Expanding along the second row, we get that $\begin{vmatrix} 3 & -2 & 4 \\ 3 & 0 & 0 \\ 0 & 8 & -3 \end{vmatrix} =$

$(3)(-1)^{2+1} \begin{vmatrix} -2 & 4 \\ 8 & -3 \end{vmatrix} + 0 + 0 =$

$-3((-2)(-3) - (8)(4)) = 78$

B3(c) Expanding along the third row, we get that $\begin{vmatrix} 0 & 1 & -2 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} =$

$0 + (1)(-1)^{3+2} \begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix} + 0 =$

$-((0)(1) - (2)(-2)) = -4$

B3(d) Expanding along the fourth row, we get that $\begin{vmatrix} 2 & 3 & -3 & 1 \\ -3 & 1 & 0 & 5 \\ 1 & 2 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{vmatrix} =$

$(3)(-1)^{4+1} \begin{vmatrix} 3 & -3 & 1 \\ 1 & 0 & 5 \\ 2 & 1 & -2 \end{vmatrix} + 0 + (1)(-1)^{4+3} \begin{vmatrix} 2 & 3 & 1 \\ -3 & 1 & 5 \\ 1 & 2 & -2 \end{vmatrix} + 0$

Let's calculate the determinants of these 3×3 matrices "on the side."

Expanding along the second row, we see that $\begin{vmatrix} 3 & -3 & 1 \\ 1 & 0 & 5 \\ 2 & 1 & -2 \end{vmatrix} =$

$(1)(-1)^{2+1} \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} + 0 + (5)(-1)^{2+3} \begin{vmatrix} 3 & -3 \\ 2 & 1 \end{vmatrix} =$

$-1((-3)(-2)-(1)(1))-5((3)(1)-(2)(-3))=-5-45=-50$

Not seeing anything clever to choose, I'll simply expand along the first row, and get that $\begin{vmatrix} 2 & 3 & 1 \\ -3 & 1 & 5 \\ 1 & 2 & -2 \end{vmatrix} =$

$2(-1)^{1+1}\begin{vmatrix} 1 & 5 \\ 2 & -2 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} =$

$2((1)(-2)-(2)(5))-3((-3)(-2)-(1)(5))+((-3)(2)-(1)(1))=-24-3-7=-34$

And so, we plug these values into our calculation for $\begin{vmatrix} 2 & 3 & -3 & 1 \\ -3 & 1 & 0 & 5 \\ 1 & 2 & 1 & -2 \\ 3 & 0 & 1 & 0 \end{vmatrix} =$

$(3)(-1)^{4+1}(-50)+(5)(-1)^{2+3}(-34)=184.$

B3(e) Expanding along the second column, we get that $\begin{vmatrix} 4 & 2 & 5 & -4 \\ -1 & 0 & 2 & 3 \\ -2 & -1 & -4 & 3 \\ 3 & 0 & -3 & 2 \end{vmatrix} =$

$(2)(-1)^{1+2}\begin{vmatrix} -1 & 2 & 3 \\ -2 & -4 & 3 \\ 3 & -3 & 2 \end{vmatrix} + 0 + (-1)(-1)^{3+2}\begin{vmatrix} 4 & 5 & -4 \\ -1 & 2 & 3 \\ 3 & -3 & 2 \end{vmatrix} + 0.$

Let's calculate the determinants of these 3×3 matrices "on the side." I'll simply use the definition of the determinant (i.e. I'll expand along the first row) as I don't see anything clever to choose.

$\begin{vmatrix} -1 & 2 & 3 \\ -2 & -4 & 3 \\ 3 & -3 & 2 \end{vmatrix} =$

$(-1)(-1)^{1+1}\begin{vmatrix} -4 & 3 \\ -3 & 2 \end{vmatrix} + (2)(-1)^{1+2}\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix} -2 & -4 \\ 3 & -3 \end{vmatrix} =$

$-1((-4)(2)-(-3)(3))-2((-2)(2)-(3)(3))+3((-2)(-3)-(3)(-4))=-1+26+54=79$

$\begin{vmatrix} 4 & 5 & -4 \\ -1 & 2 & 3 \\ 3 & -3 & 2 \end{vmatrix} =$

$(4)(-1)^{1+1}\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} + (5)(-1)^{1+2}\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} + (-4)(-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 3 & -3 \end{vmatrix} =$

$4((2)(2)-(-3)(3))-5((-1)(2)-(3)(3))-4((-1)(-3)-(3)(2))=52+55+12=119$

And so, we plug these values into our calculation for $\begin{vmatrix} 4 & 2 & 5 & -4 \\ -1 & 0 & 2 & 3 \\ -2 & -1 & -4 & 3 \\ 3 & 0 & -3 & 2 \end{vmatrix} =$ $(2)(-1)^{1+2}(79) + (-1)(-1)^{3+2}(119) = -158 + 119 = -39$

B3(f) Let's call the matrix F. Expanding along the fifth row, we get that $\det F = (-2)(-1)^{5+1} \begin{vmatrix} 3 & 4 & -5 & 7 \\ 3 & 1 & 2 & 0 \\ -1 & 4 & 1 & 0 \\ -3 & 0 & 0 & 0 \end{vmatrix} + 0 + 0 + 0 + 0.$

Expanding this 4×4 matrix along the fourth row, we get that $\det F =$

$$(-2)(-1)^{5+1} \left((-3)(-1)^{4+1} \begin{vmatrix} 4 & -5 & 7 \\ 1 & 2 & 0 \\ 4 & 1 & 0 \end{vmatrix} + 0 + 0 \right).$$

Expanding this 3×3 matrix along the third column, we get that $\det F = (-2)(-1)^{5+1}(-3)(-1)^{4+1} \left((7)(-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0 + 0 \right).$

So we have that $\det F = (-2)(1)(-3)(-1)(7)(1)((1)(1) - (4)(2)) = -42(-7) = 294$.

B4(a) Expanding along the third row of A, we get that $\det A = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 0 & 3 \\ 3 & -1 & 0 \end{vmatrix} =$

$3(-1)^{3+1} \begin{vmatrix} -1 & -2 \\ 0 & 3 \end{vmatrix} + (-1)(-1)^{3+2} \begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix} + 0 =$
$3((-1)(3) - (0)(-2)) + 1((2)(3) - (1)(-2)) = -9 + 8 = -1$

Expanding along the third column of A^T, we get that $\det A^T = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & -1 \\ -2 & 3 & 0 \end{vmatrix} =$

$(3)(-1)^{1+3} \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} + (-1)(-1)^{2+3} \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} + 0 =$
$3((-1)(3) - (-2)(0)) + 1((2)(3) - (-2)(1)) = -9 + 8 = -1$

B5(a) E_1 is an upper-triangular (and lower-triangular) matrix, and so its determinant is the product of its diagonal entries: $(1)(5)(1) = 5$

B5(b) Expanding along the first column, we get that $\det E_2 = 0+0+(1)(-1)^{3+1} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$ $1((0)(0) - (1)(1)) = -1$

B5(c) E_3 is a lower-triangular matrix, and so its determinant is the product of its diagonal entries: $(1)(1)(1) = 1$.